Robust Information Aggregation in Markets

Paris Game Theory Seminar

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Set-Up

- Investors face an investment choice $y \in \mathbb{R}_+$ with **coordination motives: The more you invest the better for me**.
- Investors have **incomplete information** about **returns** described by a common prior.
- Investors participate in a **mechanism** (implementable through market) to **aggregate their private information** prior to making investment decision.

It's a Non-Trivial Problem

- Investors participate in a **mechanism** to **aggregate their private information** prior to making investment decision.

Naive Mechanism: Public Message Board (E.g. Twitter)

- Everyone anonymously posts their private information.
- Incentive to exaggerate good news to incentivize more investment...

  Information Aggregation **not an equilibrium**.

Need something more complicated using

- **Economic Idea:** Signaling better information must be costly.

Definition, Questions and Answers

What are mechanisms that robustly aggregate agents' private information?

Informal Definition : A public signal and transfer scheme that rewards/punishes different messages s.t. **for every prior**

1. Public signal aggregates messages and publicly reveals everyone's private information
2. (Incentive Compatibility) It is in everyone's best interest to report messages leading to 1.

(1) **Can we implement it as the trading equilibrium in a separate Market?**

Existence: Yes. Agents trade a token over multiple rounds while encoding their private information into its market price. Trades implement transfers satisfying 1. and 2..

(2) **Properties of such trading equilibria?**

General Insight: As the information of players varies, prices generated by any such trading equilibrium exhibit algebraic structure generated by primes.

How to think of this exercise?

We use a separate market for tokens to implement information aggregation...

- **Positive Modeling Exercise?** Is this in part what markets are being used for? Maybe but who knows...
- **Normative Market Design/Engineering Exercise?** Is this what markets should be used for if we want to reduce information frictions? Yes.
- **Real life investment problem where such a market can be used:** Crowd-sourcing platforms e.g. Kickstarter, large scale financial investments
- **Could you implement information aggregation with non-market mechanisms?** Maybe, but we don't know how...

Class of Investment Problems

- Finite set of agents I
- Each $i \in I$ has a cash budget $b_i \in \mathbb{R}$
- Given **return parameter** $\theta \in \Theta = \mathbb{N}$, $i \in I$ chooses investment y_i s.t.

$$\max_{y_i \leq b_i} u_i(y_i, \theta, y_{-i}, b_i)$$

Class of utilities studied

- **Sufficiently Concave:** $u_i(y_i, \theta, y_{-i}, b_i)$ increasing and concave $|\partial^2 u_i| \geq K$
- **Coordination:** $(\theta, y_{-i}, b_i) \mapsto (u_i(y_i, \theta, y_{-i}, b_i) - u_i(y_i', \theta, y_{-i}, b_i))$ increasing $\forall\ y_i > y_i'$
- **Anonymity:** $u_i(y_i, \theta, y_{-i}, b_i) = u_i(y_i, \theta, y_{\kappa(-i)}, b_i)$, $\forall\ \kappa$ permutation of players

Nash Equilibrium of Investment Problem

Agents have **uncertainty** about payoff parameter $\theta \in \Theta = \mathbb{N}$

- Common priors $\mu \in \Delta(\Theta \times \prod_i S_i)$ on fixed signal space $(S_i)_i$,
- Let $X_i = \Delta(\Theta \times S_{-i})$ be the induced posteriors.

  Posterior as Signal: for every (θ, x), agent $i \in I$ privately observes posterior $x_i \in X_i$

  Use redundant notation: $\mu(\theta, x_{-i} | x_i)$

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Nash Equilibrium of Investment Problem

Given μ, budgets $b_i \in \mathbb{R}_+$ and profile of utilities $(u_i)_{i\in I}$ an investment strategy is a profile of maps $(y_i : X_i \to (-\infty, b_i])_i$, s.t.

$$y_i(x_i) \in \arg\max_{y_i \le b_i} \mathbb{E}\big(u_i(y_i, \theta, y_{-i}(x_{-i}), b_i)|x_i\big), \quad \forall\ i, x_i \in X_i$$

Class of Incomplete Information Environments

Class of Information Structures studied $\mathcal{E}$: Every prior $\mu \in \mathcal{E}$ satisfies,

1. (**No aggregate Uncertainty**) for every $(\theta, x = (x_i)_i) \in \text{supp}(\mu)$,

$$\bigcap_{i \in I} \{\hat{\theta} : \mu(\hat{\theta} | x_i) > 0\} = \{\theta\}$$

2. (**Participation**) Full support where investment outcomes of every NE under μ are equal to investment in a NE where $\theta = 0$ is common knowledge.

Road Map

Information Aggregation?

↓

Job for a communication mechanism

↓

We want a communication Mechanism that we can implement by a Market?

- imposes some restrictions on the communication mechanism

↓

Show you can achieve information aggregation in markets

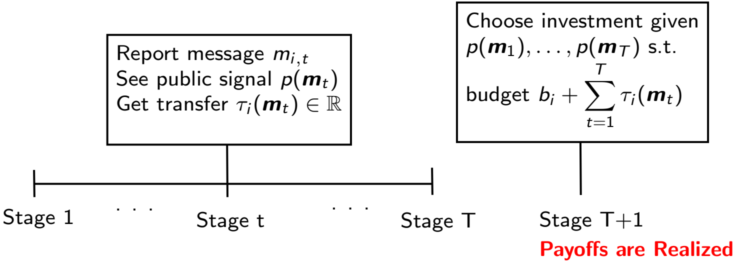
- players encode their private information into prime factorization of price

↓

Show that this is the only way of doing it that works for all information structures

Dynamic Communication Mechanism

- Messages M, Public Signal $p: M^I \to \mathbb{R}_+$, Transfers $\tau: M^I \to \mathbb{R}^I$, Stopping time T



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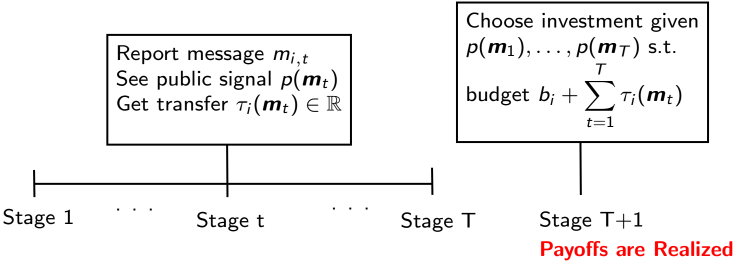

- Given history $(p(\boldsymbol{m}_1), m_{i,1}, \ldots, p(\boldsymbol{m}_{t-1}), m_{i,t-1}) \in H_i^{t-1}$, agent i chooses

1. **Reporting strategies**, $m_{i,t}: X_i \times H_i^{t-1} \to M$
2. **Investment strategy** at period $T+1$, $y_i: X_i \times H_i^T \to (-\infty, b_i + \sum_{t=1}^{T} \tau_i(\boldsymbol{m}_t)]$

Market Implementation

Implementing Dynamic Communication Mechanisms (M, p, τ, T) **with Market for Tokens:**

- endow all agents with divisible tokens
- **Messages** M represent demand/supply schedules M for tokens
- **Public signal** p is the market price of the token
- **Transfers** τ represent the token trades at given price

$\rightarrow$ Imposes market clearing and measurability requirement on transfers

Market Implementation

Dynamic Communication Mechanism (M, p, τ, T) **admits a Market Implementation** if...

(i) **Price is sufficient statistic:** Transfers only depend on m_{-i} through price:

$$\tau(m_i, m_{-i}) = \hat{\tau}(m_i, p(\boldsymbol{m}))$$

(ii) **Market Clears:**

$$\sum_i \tau_i(\boldsymbol{m}) = 0$$

From now on, restrict attention to Mech. with Market Implementation (Call them Market Mechanisms)

Market Implementation: Illustration

Market Implementation of Communication Mech: (M, p, τ, T)

Round **1**

⋮

Round **t**

⋮

Round ***T***

Round **t**

t.1 Investors **choose Demand/Supply** of assets for each price p: $m_{i,t} : p \mapsto \text{demand/supply}_{i,t}(p) \in \mathbb{R}, \quad \forall i$

t.2 Market **clearing price** is determined: $p_t \in \mathbb{R}_+$ s.t $\sum_i m_{i,t}(p_t) = 0$

t.3 Market **clearing price** p_t is publicly observed.

t.4 Trades take place at price p_t Investors leave with different endowments $b_{i,t} = b_{i,t-1} + \tau_i(m_{i,t})$

Trading stops after **endogenous** stopping time ***T*** **=** $T(p_1, \ldots, p_T)$

Investors learn by **observing prices** $p_1, \ldots, p_T$ and leave with **different endowments** $b_{i,T}$

Round ***T*+1**

Agents get utility expected utility: $\max_{y_i \in [0, b_{i,T}]} \mathbb{E}_\mu \left(u_i(y_i, \theta, y_{-i}) \;\middle|\; m_{i,1}, p_1, \ \ldots \ , m_{i,T}, p_T \right)$

Equilibrium in Augmented Investment Problem

Nash Equilibrium in **Reporting** and **Investment** Strategies

Reporting strategies, $(m_{i,t})_t$ and **Investment strategy** at period $T+1$, y_i s.t.

$$y_i(x_i, h_i^T) \in \arg\max_{y_i \le b_i^T} \mathbb{E}(u_i(y_i(x_i, h_i^T), \theta, y_{-i}(x_{-i}, h_{-i}^T), b_i^T)|x_i, h_i^T)$$

$$\text{where } b_i^T := b_i + \sum_{t=1}^{T} \tau_i(\boldsymbol{m}_t)$$

$$m_{i,t}(x_i, h_i^{t-1}) \in \arg\max_{m_{i,t} \in M} \mathbb{E}(u_i(y_i(x_i, h_i^T), \theta, y_{-i}(x_{-i}, h_{-i}^T), b_i^T)|x_i, h_i^{t-1}), \quad \forall\ t \le T$$

Robust Information Aggregation

Market Mechanism (M, p, τ, T) ε-Robustly Aggregates Information if...

$\exists$ **reporting strategy** $(m_{i,t} : X_i \times H_i^{t-1} \to M)_{i,t}$ s.t. **for every** common prior $\mu \in \mathcal{E}$,

(i) there is a NE with reporting strategy $(m_{i,t})_{i,t}$

(ii) for every draw θ, T is a **finite** stopping time satisfying

$$T = \min\{t \in \mathbb{N} : \mu(\theta | x_i, h_i^T) \geq 1 - \varepsilon\}$$

Market Mechanism (M, p, τ, T) Robustly Aggregates Information if...

it ε-Robustly Aggregates Information for all ε small enough.

Existence of Robust Aggregation

Lemma 1 (Existence)

There exists a Market mechanism that robustly aggregates information.

Construction in Simple Example: Uniform Beliefs

- $\Theta = \mathbb{N}$, $\mu(\theta) > 0$, $\forall\ \theta \in \Theta$,
- Three agents i_1, i_2, i_3.
- Each i observes noisy signal $s_i = \theta + \epsilon_i$,

Signals given θ

Signal Probability $\mathbf{i_1}$	1/3		1/3		1/3
Signal set $\mathbf{i_1}$	$\theta - 2$		θ		$\theta + 2$
Signal Probability $\mathbf{i_2}$		1/3	1/3	1/3	
Signal set $\mathbf{i_2}$		$\theta - 1$	θ	$\theta + 1$	
Signal Probability $\mathbf{i_3}$		1/3	1/3		1/3
Signal set $\mathbf{i_3}$		$\theta - 1$	θ		$\theta + 2$

Trading Protocol: Uniform Beliefs

- Endow every agent with infinitely many tokens,
- Identify Θ with prime numbers, where $\theta \mapsto P_\theta$, the θ-th prime
- **Message Space**: Token demand/supply schedule
  $M = \{m \colon p \mapsto \text{ token demand/supply at price } p\}$,
- **Round 1:**

$$m_i(p, s_i) = \underbrace{\frac{\alpha}{p} \sum_{\theta : \mu(\theta|s_i) > 0} \log P_\theta}_{\text{Demand}} - \underbrace{1}_{\text{(inelastic) supply}}, \quad \alpha > 0$$

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- **Market clearing**, $p(\boldsymbol{m}, s)$ is p s.t. $\sum_i m_i(p, s_i) = 0$,

Trading Protocol: Uniform Beliefs

For $\theta > 2$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ we have Θ-information sets:

Information set i_1	$\theta - 2$		θ		$\theta + 2$
Information set i_2		$\theta - 1$	θ	$\theta + 1$	
Information set i_3	$\theta - 2$		θ	$\theta + 1$	

$$\frac{p(\boldsymbol{m})}{\alpha} = \frac{2 \log P_{\theta-2} + \log P_{\theta-1} + 3 \log P_{\theta} + 2 \log P_{\theta+1} + \log P_{\theta+2}}{3}$$

$$e^{\frac{3}{\alpha} p_1(\boldsymbol{m})} = (P_{\theta-2})^2 \cdot P_{\theta-1} \cdot (P_{\theta})^3 \cdot (P_{\theta+1})^2 \cdot P_{\theta+2}$$

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- **Prime factorization** of $e^{\frac{3}{\alpha} p}$ reveals θ as prime with **largest exponent**
- **Computationally** easy for participants, very hard for outsiders.

Construction in Simple Example

What about **non-uniform** beliefs?

Construction in Simple Example: General Beliefs

- $\Theta = \mathbb{N}$, $\mu(\theta) > 0$, $\forall\ \theta \in \Theta$,
- Three agents i_1, i_2, i_3.
- Each i observes noisy signal $s_i = \theta + \epsilon_i$,

Signals given θ

Signal Probability $\mathbf{i_1}$	$q_{i_1}^{\theta-2}$		$q_{i_1}^{\theta}$		$q_{i_1}^{\theta+2}$
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Signal Probability $\mathbf{i_3}$		$q_{i_3}^{\theta-1}$	$q_{i_3}^{\theta}$		$q_{i_3}^{\theta+2}$
Signal set $\mathbf{i_3}$		$\theta - 1$	θ		$\theta + 2$

Assume there is $n \in \mathbb{N}$ s.t. $\forall i$, $\hat{\theta}$, $\mu(\hat{\theta}|s_i) \in \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$

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- **Market clearing**, $p_1(\boldsymbol{m}_1, s)$ is p_1 s.t. $\sum_i m_{i,1}(p_1, s_i) = 0$,

Trading Protocol: Round 1

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Define Aggregate Belief Coefficient: $\kappa_\theta^1 := n \cdot \sum_{i \in I} \mu(\theta | s_i) \in \mathbb{N}$.

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$$\frac{p_1(\boldsymbol{m})}{\alpha} = \frac{\kappa^1_{\theta-2} \log P_{\theta-2} + \kappa^1_{\theta-1} \log P_{\theta-1} + \kappa^1_\theta \log P_\theta + \kappa^1_{\theta+1} \log P_{\theta+1} + \kappa^1_{\theta+2} \log P_{\theta+2}}{3n}$$

$$e^{\frac{3n}{\alpha} p_1(\boldsymbol{m})} = (P_{\theta-2})^{\kappa^1_{\theta-2}} (P_{\theta-1})^{\kappa^1_{\theta-2}} (P_\theta)^{\kappa^1_\theta} (P_{\theta+1})^{\kappa^1_{\theta+1}} (P_{\theta+2})^{\kappa^1_{\theta+2}}$$

- **Prime factorization** of $e^{\frac{3n}{\alpha} p_1}$ reveals union of information sets and κ^1

Trading Protocol: Rounds $t > 1$

- Need to repeat procedure over multiple rounds: Given histories h^{t-1},

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$$m_{i,t}(p_t, s_i, h_i^{t-1}) = \underbrace{\frac{\alpha}{p_t} \sum_{\theta' : \hat{\mu}(\theta' | s_i, h_i^{t-1}) > 0} \hat{\mu}(\theta' | s_i, h_i^{t-1}) \cdot \log P_{\theta'}}_{\text{Demand}} - \underbrace{1}_{\text{(inelastic) supply}}$$

Market Clearing: $e^{\frac{3np_t}{\alpha}} = (P_{\theta-2})^{\kappa^t_{\theta-2}} (P_{\theta-1})^{\kappa^t_{\theta-2}} (P_\theta)^{\kappa^t_\theta} (P_{\theta+1})^{\kappa^t_{\theta+1}} (P_{\theta+2})^{\kappa^t_{\theta+2}}$

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Proposition: Aggregate Belief Coefficients Converge

$(\kappa^t)_t$ (sum of beliefs) converge as $t \uparrow \infty$.

Trading Protocol: Final Round

- **Stop** this procedure as soon as $||\kappa^{t-1} - \kappa^{t}||_\infty < \epsilon/n$
- Final round T: Let $\Theta(p_{T-1}) := \bigcup_{i \in I} \{\theta' : \mu(\theta'|x_i) > 0\}$

$$m_{i,T}(p_T, s_i, h_i^{T-1}) = \underbrace{\frac{\alpha}{p_T} \sum_{\theta' : \hat{\mu}(\theta'|s_i, h_i^{T-2}) > 0} \hat{\mu}(\theta'|s_i, h_i^{T-2}) \cdot \log P_{\theta'}}_{\text{Demand}}$$

$$\underbrace{-1 - \frac{\alpha}{p_T} \sum_{\theta' \in \Theta(p_{T-1}) : \hat{\mu}(\theta'|s_i, h_i^{T-1}) = 0} \frac{\kappa_{\theta'}^{T-1} \log P_{\theta'}}{n}}_{\text{supply}}$$

Market Clearing: $e^{\frac{3np_T}{\alpha}} = \frac{(P_\theta)^{\kappa_\theta^{T-1}}}{\prod_{\hat{\theta} \neq \theta} (P_{\hat{\theta}})^{k_{\hat{\theta}}^{T-1}}}$

Trading Protocol: General Beliefs

- **Prime Factorization gives union of information sets:** Given $e^{3np/\alpha}$ we can recover the set of prime numbers factoring it.

- Bayesian Learning if everyone follows protocol:

  The state θ is the unique prime largest (positive) exponent in factorization of $e^{3np_T/\alpha}$

  Off Eq.-path Learning? Under a deviation such a prime may not exist, in which case we assume agents do not update beliefs.

Discouraging Deviations

- **Is the trading protocol an equilibrium for this prior?** Argument for uniform beliefs:

- **Fact (Bertrand–Chebyshev theorem):** $\log P_m - \log P_{m-1} \leq \log 2$.

  Local Deviations for i:

  1. Reporting information set realized under $\theta + 1$,
     Monetary Cost: At least $\alpha \log 2$
     Monetary Benefit: At most $y^*_{-i}(\theta + 1) - y^*_{-i}(\theta)$ if all players learn $\theta + 1$

  2. Reporting information set realized under $\theta - 1$,
     Monetary Cost: At least $y_{-i}(\theta) - y^*_{-i}(\theta - 1)$ if information aggregation succeeds
     Monetary Benefit: At most $\alpha \log 2$

- **Globally bounded concavity:** $\exists\ \alpha$ discouraging deviations locally, then implies no global deviations.

Construction in General

What features made this example so simple?

- Infinitely supported beliefs $\rightarrow$ ε aggregation
- Many players with same information $\rightarrow$ Repeated mixed strategy until information aggregation succeeds
- Signals are not identified by θ, i.e. there is no complete order on state space $\Theta \times X$ representing preferences of <u>all</u> agents, $\rightarrow$ One round for every ordered subset

Relaxing any of them requires multiple trading rounds but idea is the same: **Aggregation achieved by players jointly controlling prime factorization of market price**.

Algebraic Structure of Equilibria

Interesting Information Structures $\mathcal{E}^* \subseteq \mathcal{E}$

- **Need at least 3 players to pin down state:** There is $\delta > 0$ so that for all pairs $i_1, i_2 \in I$, $\mu(\theta | x_{i_1}, x_{i_2}) < 1 - \delta$

Algebraic Structure of Equilibria

A binary operation $\oplus$ on priors $\mathcal{E}^* \subseteq \mathcal{E}$ is monotonic if for any priors $\mu_1, \mu_2 \in \mathcal{E}^*$,

(i) **Better News:** $\forall\ \ell \in \{1,2\},\ \forall\ x_i, \forall\ i,$

$$\text{marg}_\Theta \mu_\ell(\cdot|x_i) \preceq_{\text{FOSD}} \text{marg}_\Theta(\mu_1 \oplus \mu_2)(\cdot|x_i),\ \text{strict for some agent } i$$

Generator

A generator of $(\mathcal{E}^*, \oplus)$ is a subset $E \subseteq \mathcal{E}^*$ so that every $\mu \in \mathcal{E}^*$ can be written as a finite sum $\oplus$ of elements in E.

Algebraic Structure of Equilibria

Fix a Market mechanism, with robust IA-equilibrium $\sigma = (m, y)$ so that for all other such equilibria σ' and all priors μ, $\mathbb{E}_\mu(T(\theta, x, \sigma)) \leq \mathbb{E}_\mu(T(\theta, x, \sigma'))$.

- σ induces a mapping from common priors to observable prices in every round:

$$p_{t,\sigma} : \mathcal{E}^* \to \mathcal{P}_t \subseteq \mathbb{R}_+$$

- Every monotonic binary operation on priors $\oplus$ induces a binary operation $\otimes$ on price histories:

$$p_{t,\sigma}(\mu_1) \otimes p_{t,\sigma}(\mu_2) := p_{t,\sigma}(\mu_1 \oplus \mu_2)$$

Theorem 1: Prime generator

For every ε-robust information aggregation equilibrium σ and round t, there is a countable set $\mathbb{P}_t \subseteq \mathcal{P}_t$ s.t. for every monotonic $\oplus$ and every $\mu \in \mathcal{E}^*$, $p_{t,\sigma}(\mu)$ can be uniquely written as a finite $\otimes$-product of elements in $\mathbb{P}_t$.

Explaining the Statement and Proof Idea

- Robust information aggregation must consist of controlling the prime factorization of the price

  Why? **Two Steps**

1. There could be multiple players, each with something unique to say which needs to be recovered from the price:

- **Uniqueness property:** It should not be possible to write her message as a combination of other agent's message (avoid confounding)
- Primes represent a minimal way of encoding information in a robust way.

2. Why can we only use this minimal way in Equilibrium?

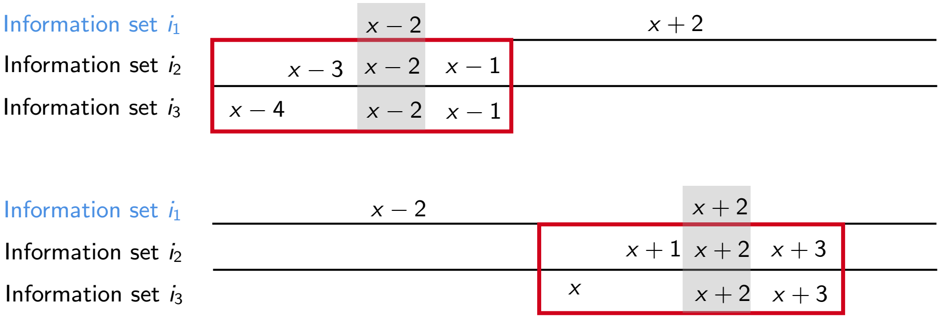
- **Minimality property:** Reporting is costly, if there are two ways of encoding the same information, only use the cheapest.

Conclusion

- Robust information aggregation through a separate token market is possible
- Agents can only do it by jointly controlling prime factorization of market price of the token

Appendix: Beliefs

Second Order Information Sets of i_1



Appendix: Discouraging Deviations

Deviations of i_1