Strategic Type Spaces

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Universal Representation of Payoff-Relevant Info

- Describe **all** players' information:

Harsanyi type spaces (HTS) $\rightarrow$ Universal Type Space (UTS)

- UTS $\cong$ hierarchies of beliefs: very big space, not big enough for all solution concepts
- Fixed **game** and **solution concept** (Rationalizability):

What is a **minimal** universal representation?

Universal Representation of Payoff-Relevant Info

What we do:

- **Existence** + **Uniqueness** of minimal universal representation for finite game and rationalizability $\cong$ hierarchies of best-replies
- Show it has **recursive** structure on a **finite** set.

Information Structure

- Finite set of States of Nature: K
- Finite set of players: I

Harsanyi Type Space (HTS), $(T_i, \pi_i)_{i \in I}$

- A topological space of types $\forall\ i \in I,\ \ T_i,\ \ T_{-i} := \prod_{j \neq i} T_j$
- continuous map: $\forall\ i \in I,\ \pi_i \colon T_i \to \Delta(K \times T_{-i})$ with weak topology on $\Delta(K \times T_{-i})$

Universal Type Space

Universal Type Space (UTS), (Mertens and Zamir 85)

HTS $(T_i^*, \pi_i^*)_i$ so that for every HTS $(T_i, \pi_i)_i$ and player i there exist continuous maps $\xi_i \colon T_i \to T_i^*$ so that the diagram below commutes

$$
\begin{array}{ccc}
T_i & \xrightarrow{\pi_i} & \Delta(K \times T_{-i}) \\
\Big\downarrow \xi_i & & \text{id} \Big\downarrow \quad \Big\downarrow \xi_{-i} \\
T_i^* & \xrightarrow{\pi_i^*} & \Delta(K \times T_{-i}^*)
\end{array}
$$

Lemma 1, (Mertens and Zamir 85)

- (Existence and Uniqueness) UTS exists and is unique up to homeomorphisms
- (Characterization) HTS $(T_i^*, \pi_i^*)_i$ is UTS if and only if for every i, π_i^* is a homeomorphism

Type Space Quotients

Type Space Quotient (TSQ)

A TSQ $(S_i, \psi_i)_{i \in I}$, where for every i, S_i is a topological space and for every HTS $(T_i, \pi_i)_i$ there exist continuous map $\eta_i \colon T_i \to S_i$ so that

$$
\begin{array}{ccc}
T_i & \xrightarrow{\pi_i} & \Delta(K \times T_{-i}) \\
\downarrow \eta_i & & \text{id}\downarrow \quad \downarrow \eta_{-i} \\
S_i & \xleftarrow[\psi_i]{} & \Delta(K \times S_{-i})
\end{array}
$$

Verbal Explanation:

η_i coarsens types and beliefs of player i into equivalence classes S_i

ψ_i: many beliefs correspond to same s_i, (UTS had homeomorphism)

Diagram: If beliefs of t_i, t_i' on $K \times \eta_{-i}(T_{-i})$ coincide then $\eta_i(t_i) = \eta_i(t_i')$

Fix Game and Solution Concept

- Fix finite **action** sets $(A_i)_i$, **payoffs** $u\colon K \times \prod_i A_i \to \mathbb{R}^I$
- **Interim Correlated Rationalizability**: $ICR_i : T_i \to 2^{A_i}$ so that for every $t_i \in T_i$
  $ICR_i(t_i)$ is the set of **best-replies** to beliefs of the form

$$p(k, a_{-i}) = \int_{K \times T_{-i}} \sigma(a_{-i}|k, t_{-i})\ \pi_i(k, \mathrm{d}t_{-i}|t_i)$$

  for any **random selection** $\sigma(\cdot|k, t_{-i}) \in \Delta(ICR_{-i}(t_{-i}))$ for all t_{-i}

Best-Response Map

Best-Response map, $(\mathrm{BR}_i)_{i \in I}$

- $\forall\ i \in I$, $\mathcal{A}_i := 2^{A_i}$
- Best-response map: $\mathrm{BR}_i \colon \Delta(K \times \mathcal{A}_{-i}) \to \mathcal{A}_i$ s.t. $\mathrm{BR}_i^{-1}(a_i)$ convex $\forall\ a_i \in \mathcal{A}_i$

Lemma 2: ICR as Fixed Point of a Best-Response map

For every finite action sets $(A_i)_i$, payoffs $u : K \times \prod_i A_i \to \mathbb{R}^I$, every player i there is a best response map BR_i with $\mathcal{A}_i = 2^{A_i}$ so that for every HTS (T_i, π_i),

$$
\begin{array}{ccc}
T_i & \xrightarrow{\pi_i} & \Delta(K \times T_{-i}) \\
\Big\downarrow {\scriptstyle ICR_i} & & {\scriptstyle \mathrm{id}}\Big\downarrow \quad \Big\downarrow {\scriptstyle ICR_{-i}} \\
\mathcal{A}_i & \xleftarrow[\mathrm{BR}_i]{} & \Delta(K \times \mathcal{A}_{-i})
\end{array}
$$

Strategic Type Space

BR-Orbit (for ICR)

A BR-orbit on a TSQ $(S_i, \psi_i)_{i \in I}$ is a sequence $(\sigma^m)_{m=0,1,\ldots}$ s.t. $\forall \; i$, $\sigma_i^0(S_i) = \{A_i\}$ is constant and for all m,

$$\begin{array}{ccc} \Delta(K \times S_{-i}) & \xrightarrow{\psi_i} & S_i \\ \text{id}\downarrow \quad \downarrow\sigma^m_{-i} & & \downarrow\sigma_i^{m+1} \\ \Delta(K \times \mathcal{A}_{-i}) & \xrightarrow{\mathrm{BR}_i} & \mathcal{A}_i \end{array}$$

- S_i must be rich enough so that BR_i is measurable for all σ^m_{-i}

Strategic Type Space (for ICR)

The STS is the coarsest TSQ $(S_i, \psi_i)_{i \in I}$ that admits a BR-orbit starting with constant map $\sigma^0 = \{A\}$.

Existence and Uniqueness

Let $(S_i, \psi_i)_{i \in I}$ be a TQS admitting a BR-orbit $(\sigma^m)_{m=0,1,\ldots}$,

- Define set of BR-hierarchies $S_i^* := \{(\sigma^m{}_i(s_i))_m : s_i \in S_i\} \subseteq \mathcal{A}^{\mathbb{N}}$
- $\forall\, p \in \Delta(K \times S_{-i}^*)$ define the map $\psi_i^* \colon \Delta(K \times S_{-i}^*) \to S_i^*$

$$\psi_i^*(p) := (\mathrm{BR}_i(\mathrm{marg}_{K,m}(p)))_m$$

Lemma 3: Existence and Uniqueness

- $(S_i^*, \psi_i^*)_i$ is a Strategic Type Space and every STS is homeomorphic to it.
- There is $\beta \colon S^* \to \mathcal{A}$ s.t. for every HTS (T, π), $\beta \circ \eta^*(t) = ICR(t)$.

Finite Recursive Representation

Let $(S_i^*, \psi_i^*)_{i \in I}$ be the STS, the shift operator, **shift**: $(a^1, a^2, a^3, \dots) \mapsto (a^2, a^3, \dots)$

For every $\boldsymbol{a}^m = (a^1, \dots, a^m) \in \mathcal{A}^m$, define truncated orbits

$$\mathcal{A}^m(S, \boldsymbol{a}^m) := \left\{(s^n)_{n \geq m} : s \in S^* \text{ s.t. } s^n = a^1, \dots, s^m = a^m\right\} \subseteq \mathcal{A}^{\mathbb{N}}$$
$$\mathcal{A}(S) := \{\mathcal{A}(S, \boldsymbol{a}^m) : \boldsymbol{a}^m \in \mathcal{A}^m,\ m \in \mathbb{N}\}$$

Define binary relation on truncated orbits: $\forall \boldsymbol{A}, \boldsymbol{A}' \in \mathcal{A}(S)$,

$$\boldsymbol{A} \preceq \boldsymbol{A}' \iff \boldsymbol{A} \subseteq \{\textbf{shift}(\boldsymbol{a}') : \boldsymbol{a}' \in \boldsymbol{A}'\}$$

Theorem 1: Finite Recursive Representation of STS

- The set of truncated orbits $(\mathcal{A}(S), \preceq)$ is finite.
- S^* is homeomoprhic to the set of $\preceq$-monotone sequences on $\mathcal{A}(S)$

Conclusion

- Proved **Existence + Uniqueness** of minimal universal representation of payoff-relevant information.
- Showed that the space is generated through **recursion** by a binary relation on a **finite set** of truncated orbits.
- **Comparison to UTS**: UTS can only be generated through recursion on the UTS itself!

Appendix

Appendix

Construction of BR-hierarchies

BR-hierarchies

$$S_i^1 = \{A_i^0\}$$

$$S_i^2 = \{\mathrm{BR}_i(p) : p \in \Delta(K \times S_{-i}^1)\}$$

$\vdots$

$$S_i^m = \{(\mathrm{BR}_i(p_n))_{n \leq m} : \exists\, p \in \Delta(K \times S_{-i}^{m-1}) \text{ s.t. } \forall n < m, p_n = \mathrm{marg}_{K,n}(p)\}$$

$\vdots$

Belief-hierarchies

$$T_i^1 = \Delta(K)$$

$$T_i^2 = \{(t_i^1, t_i^2) \in T_i^1 \times \Delta(K \times T_{-i}^1) : \mathrm{marg}_K(t_i^2) = t_i^1\}$$

$\vdots$

$$T_i^{m+1} = \{(t_i^m, t_i^{m+1}) \in T_i^m \times \Delta(K \times T_{-i}^m) : \mathrm{marg}_K(t_i^2) = t_i^1\}$$

$\vdots$

Strategic Type Space

For all $p \in \Delta(K \times S_{-i}^{m-1})$, define $\psi^m(p) := (\mathrm{BR}_i(\mathrm{marg}_{K,n}(p)))_{n<m}$

Lemma 4: Strategic Type Space

- For every i the sequence $(S_i^m, \psi_i^m)_m$ extends uniquely to a limit (S_i^*, ψ_i^*) s.t. $\forall$ m,

$$\begin{array}{ccc} S_i^* & \xleftarrow[\psi_i^*]{} & \Delta(K \times S_{-i}^*) \\ \Big\downarrow \mathrm{proj}_m & & \mathrm{id} \Big\downarrow \quad \Big\downarrow \mathrm{proj}_{m-1} \\ S_i^m & \xleftarrow[\psi_i^m]{} & \Delta(K \times S_{-i}^{m-1}) \end{array}$$

- Limit profile $(S_i^*, \psi_i^*)_i$ is a minimal Strategic Type Space
- Every other minimal STS is homeomorphic to $(S_i^*, \psi_i^*)_i$
- The BR-orbit $(\sigma^m)_m$ is given by coordinate projections.

Coarsest Type Space Quotient

Coarser Type Space Quotient

A STS $(S_i^*, \psi_i^*)_{i \in I}$ is **coarser** than $(S_i, \psi_i)_{i \in I}$ if for every i there exist continuous surjection $\zeta_i \colon S_i \to S_i^*$ so that

$$\begin{array}{ccc} S_i & \xleftarrow[\psi_i]{} & \Delta(K \times S_{-i}) \\ \Big\downarrow{\scriptstyle \zeta_i} & & {\scriptstyle \mathrm{id}}\Big\downarrow \quad \Big\downarrow{\scriptstyle \zeta_{-i}} \\ S_i^* & \xleftarrow[\psi_i^*]{} & \Delta(K \times S_{-i}^*) \end{array}$$