Strategic Type Spaces and Robustness to Incomplete Information*

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Abstract

Given any game with incomplete information we define strategic type spaces (STS) as representations of players strategically relevant information. We prove existence and essential uniqueness of a minimal STS, and that this minimal STS is a quotient of the universal type space. We show that the minimal STS admits a finite representation. On common prior models, we characterize rationalizable strategies that are robust to incomplete information in terms of a finite system of polynomial equations derived from the finite representation of the STS.

1 Introduction

For games of incomplete information, Harsanyi (1967) introduced type spaces as models to describe players' information on uncertain payoff-relevant parameters (i.e. states of nature). A type is associated to a belief on states of

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nature and other players' types. Mertens and Zamir (1985) show that these type spaces can be represented in canonical models (universal type spaces) of players' hierarchies of beliefs, whose descriptions do not dependent on the payoff structure of a game. Universal types should then contain payoff relevant information for all games and so become very intractable objects.

This paper takes back the question of how to describe players' information by taking a game (or a class of games) as fixed. We provide a universal representation of players' payoff relevant information by switching the focus from a purely informational description of types to a strategic description. For a fixed game, we introduce strategic type spaces (STS) as strategic descriptions of players' information, which admit an economical representation of every Harsanyi type space.

In our approach, an economical representation of a Harsanyi type space is obtained by mapping its types and associated beliefs into equivalence classes. A STS is a canonical set of such equivalence classes, called strategic types, which satisfies the following two conditions:

1. For any Harsanyi type space, if two types' beliefs coincide on STS, then these types are mapped to the same strategic type.

2. STS admits measurable best reply maps to any strategic behavior of players' strategic types.

The first is a sufficient condition for different Harsanyi types to be merged into the same strategic type. Unlike Harsanyi (1967), we do not require the converse of this condition. We thus allow for different beliefs over states and other players' strategic types to correspond to the same strategic type. This allows for types to partition beleifs. The second condition depends on a best-reply concept and the meaning of strategic behavior. We focus on the best-reply correspondence that underlies the solution concept of interim correlated rationalizability (ICR), as defined in Dekel, Fudenberg, and Morris (2007). In this setting, a type's strategic behavior is a set of correlated strategies.

A STS is minimal if, among all STS, it admits the coarsest representation of every Harsanyi type space.

We provide an axiomatization of STS from the conditions described above and use it to express our notion of minimality. We prove existence and uniqueness of the minimal STS. We show this by first proving that all finite order ICR actions arising from any Harsanyi type can be recovered from any STS. We then provide a canonical construction of the set of best-reply hierarchies and show that it forms a STS. We show that these hierarchies characterize all finite order ICR actions and deduce that our construction characterizes the minimal STS.

We then provide a finite representation of the minimal STS. For each finite order best-reply hierarchy we define its z-state as the collection of action sets it reached and, for each such action set the number of times it revisited this action set modulo z. We show that there is bounded z so that the set of all possible transitions from an m-order best-reply hierarchy to an $m+1$ order best-reply hierarchy depend only on its z-state. We deduce that the minimal STS is canonically isomorphic to the orbit of an operator on the finite set of z-states. We call this the STS-automaton.

We adopt the concept of robustness to incomplete information for complete information games introduced in Kajii and Morris (1997). A prediction of a complete information solution concept is robust to incomplete information if it is also a prediction in every nearby common prior model. Kajii and Morris (1997) and Morris and Ui (2005) provide sufficient conditions for robustness of Nash equilibria and Oyama and Takahashi (2020) provide a characterization of robustness to incomplete information of Nash equilibria for supermodular games. We provide a characterization of robust rationalizable actions for any finite complete information game in terms of a finite system of polynomial equations, which we derive from the STS automaton. We proceed as follow: First we show that any nearby common prior model of a given complete information game admits a representation as a nearby common prior on its STS. We then show that for any common prior on an STS that is close to a given complete information game, there is an even closer common prior on an STS with interim beliefs which are measurable with respect to the z-states of the associated STS automaton. We show that marginal probabilities on types of such common priors exhibit a finite recursive structure which is characterized by a finite system of polynomials derived from the STS-automaton. In particular, we show that a system of interim beliefs on z-states is consistent with a z-state-measurable common prior if and only if the associated finite system of polynomials admits a solution where each unknown is in $(0,1)$. We deduce that robustness is characterized by a set of solutions of a convex family of systems of polynomials. We show that this family admits finitely many extreme points and thus obtain a characterization of robustness in terms of a finite system of polynomial equations for any finite game.

The main part of the paper is organized in three main parts:

> In Section 3 we introduce Strategic type spaces (STS). Section 3.1 introduces the best-reply correspondence underlying the concept of Interim correlated rationalizability. In section 3.2 we introduce STS and minimal STS axiomatically. In section 3.3 we establish existence and uniqueness of a minimal STS, characterized as the space of best-reply hierarchies.
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> In Section 4 we show that the minimal STS can be represented by a finite automaton and exhibits a recursive structure of the STS. We provide some illustrative examples where we construct the STS automaton and thus the minimal STS for a two player game.
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> Finally, in Section 5 we provide a characterization of rationalizable strategies which are robust to incomplete information for all finite complete information games.

1.1 Related Literature

The best-reply concept we focus on in this paper was introduced to define Interim correlated rationalizability (ICR). Rationalizability was introduced by Bernheim (1984); Pearce (1984) in games with complete information. Dekel et al. (2007) generalized this concept by introducing the concept of ICR for games of incomplete information. For every type, ICR iteratively eliminates never best-replies to that type's expectation over any state contingent, correlated beliefs over other types' actions.

Dekel et al. (2007) show that two Harsanyi types have the same ICR actions in all games if and only if they correspond to the same hierarchy of beliefs and hence the same point in the universal type space of Mertens and Zamir (1985). Therefore, ICR has been studied as a correspondence on the universal type space of Mertens and Zamir (1985). Morris, Shin, and Yildiz (2016) characterizes ICR in terms of a common belief operator on the universal type space for global games. Weinstein and Yildiz (2007) first identified critical types, i.e. points of discontinuity of ICR in the universal type space of Mertens and Zamir (1985). They provide a topological characterization of critical types. Dekel, Fudenberg, and Morris (2006) and Chen, Di Tillio, Faingold, and Xiong (2016a) characterize the coarsest topology on the universal type space, called the strategic topology, under which ICR is continuous. Chen et al. (2016a) introduce the notion of *frames* as partitions of the universal type space similar to the first property of STS. They use frames as a tool to define a strategic topology of uniform convergence over games for hierarchies of beliefs. Chen, Takahashi, and Xiong (2014) study robustness of ICR to both higher order beliefs and payoff perturbations. The authors define *curb* collections which is closely related to the second requirement of STS, i.e. strategic closure (see Section 3.2), defined in terms of the universal type space. Chen, Takahashi, and Xiong (2016b) provide an algorithm to compute hierarchies of ICR which parallels our construction of best reply hierarchies. Based on their construction, they study refinements on ICR. Finally, Ely and Peski (2011) provide a characterization of critical types in terms of common belief properties in the universal type space.

This paper differs from the literature described above in the following way: We fix a game and introduce a canonical language to describe strategically relevant information for this game. Unlike frames and curb collections, STS are defined as universal objects which can be characterized and constructed without reference to Harsanyi types. Moreover, the characterization of critical types in terms of strategic complexity relies on the coarser structure obtained by studying ICR through the lens of STS.

On common prior models Kajii and Morris (1997) and Morris and Ui (2005) provide sufficient conditions for robustness of Nash equilibrium and Oyama and Takahashi (2020) provide a characterization of robustness to incomplete information of Nash equilibria for supermodular games. Both papers rely on so called potential functions to obtain their results. We use the STS and provide a characterization of robustness of rationalizable strategies for all finite games.

2 Preliminaries and Notations

We denote the cardinality of a set Y by $\#Y$ and let $\#\varnothing = 0$. For a family of sets $(X_i)_i$ we let $X := \prod_i X_i$ and $X_{-i} := \prod_{j\neq i} X_j$; the disjoint union is denoted $\coprod_i X_i$. We denote the i-th coordinate projection by proj_i. For a family of mappings $f_i\colon X_i \to Y_i$, f is the map from X to Y given by $f(x) = (f_i(x_i))_i$ and f_{-i} is from X_{-i} to Y_{-i} is given by $f_{-i}(x_{-i}) = (f_j(x_j))_{j\neq i}$. Similarly, if $f\colon X \to Y$ and $g : Z \to W$ are mappings we denote by $(f \times g) : X \times Z \to Y \times W$ the map given by $(f \times g)(x, z) = (f(x), g(z))$. On any set X we denote by id_X the identity mapping on X and omit the subscript X when there is no ambiguity. The set of Borel probability measures over a topological space X is written as Δ_X. We denote by supp p the support of a probability distribution p. The marginal probability on X_m of a probability measure p on a product space $X = \prod_i X_i$ is $\mathrm{marg}_m(p)$.

In commutative diagrams we describe a mapping between probability measures from Δ_X to Δ_Y which are induced by a measurable mapping from X to Y by an arrow on the subscripts as follows:

$$\begin{array}{c} \Delta_X \\ \downarrow \\ \Delta_Y \end{array}$$

Double arrows such as $X \rightrightarrows Y$ represent correspondences, hooked arrows such as $A \hookrightarrow X$ represent embeddings (i.e. maps which are isomorphisms onto their image space), and double headed arrows such as $X \twoheadrightarrow Y$ denote surjective mappings. A dashed arrow such as $X \dashrightarrow Y$ denotes a mapping which is defined by commutativity of the diagram.

The subscript i denotes a typical player from the finite set N of players. A finite set K of states of nature and, finite action sets $(A_i)_{i\in N}$ and a payoff function $u : A \times K \to \mathbb{R}^N$, are given.

3 Strategic Type Spaces

3.1 Interim Correlated Best-Replies

Dekel et al. (2007) show that ICR can be defined as a fixed point of a best reply correspondence, which we now state.

We introduce the set $\mathcal{B}_i := 2^{A_i}$ of action subsets and define conjectures as maps $\sigma : K \times \mathcal{B}_{-i} \to \Delta(A_{-i})$. The probability $\langle \sigma, p\rangle$ over $K \times A_{-i}$ induced by a belief $p \in \Delta_{K\times\mathcal{B}_{-i}}$ and a conjecture σ given by the relation:

$$\langle \sigma, p\rangle(k, a_{-i}) := \sum_{b_{-i}\in\mathcal{B}_{-i}} \sigma(k, b_{-i})(a_{-i})\; p(k, b_{-i}),$$

and player i's best-reply correspondence $\mathrm{BR}_i : \Delta_{K\times\mathcal{B}_{-i}} \rightrightarrows \mathcal{B}_i$ is given by:

$$\mathrm{BR}_i(p) := \left\{ \arg\max_{a_i\in A_i} \sum_{k,a_{-i}} u_i(a_i, a_{-i}, k)\langle \sigma, p\rangle(k, a_{-i}) : \mathrm{supp}\ \sigma(k, b_{-i}) \subseteq b_{-i} \right\}.$$

A *Harsanyi type space* $\mathfrak{H}$ consists of a family of topological spaces $(\Theta_i)_{i\in N}$ and of continuous mappings $\pi_i : \Theta_i \to \Delta_{K\times\Theta_{-i}}$, where $\pi_i(\theta_i)$ represents type θ_i's belief over types of other players and states of nature. As in Dekel et al. (2007), we rely on the best-reply correspondence BR_i to define ICR on any Harsanyi type space $(\Theta_i, \pi_i)_i$ as follows: ICR of a type θ_i is given by $\mathrm{ICR}_i(\theta_i) = \bigcap_m \mathrm{ICR}_i^m(\theta_i)$, where $\mathrm{ICR}_i^0(\theta_i) = A_i$ and $\mathrm{ICR}_i^m(\theta_i)$ is i's best response to the $\pi_i(\theta_i)$-mixtures (i.e. an expectation $\int_{\Theta_{-i}} \sigma(k,\theta_{-i})\pi_i(\theta_i)(k, \mathrm{d}\theta_{-i})$) of all measurable, state and type profile contingent conjectures $\sigma(k,\theta_{-i}) \in \Delta(A_{-i})$ whose support is contained in $\mathrm{ICR}_{-i}^{m-1}(\theta_{-i})$ for all θ_{-i}. We call the sequence $(\mathrm{ICR}^m(\theta))_{m\geq 0}$ the ICR-*hierarchy* of θ.

3.2 Strategic Type Spaces

We define a strategic type space (for ICR) as a pair $\mathfrak{S} = (\mathcal{S}_i, \psi_i)_i$ consisting of an N-tuple of topological spaces $\mathcal{S}_i$ and continuous maps $\psi_i : \Delta_{K\times\mathcal{S}_{-i}} \to \mathcal{S}_i$ which satisfy both a type space quotient requirement and a strategic requirement.

Definition 3.1 (Type Space Quotient). *A space* $\mathfrak{S} = (\mathcal{S}_i, \psi_i)_i$ *is a* Type Space Quotient *if, for every Harsanyi type space* $\mathfrak{H} = (\Theta_i, \pi_i)_i$ *there exist a family of maps* $(\eta_i)_i$ *for which the following diagram commutes:*

$$\begin{array}{ccc} \Theta_i & \xrightarrow{\pi_i} & \Delta_{K\times\Theta_{-i}} \\ \Big\downarrow \eta_i & & id \Big\downarrow \;\; \Big\downarrow \eta_{-i} \\ \mathcal{S}_i & \xleftarrow[\psi_i]{} & \Delta_{K\times\mathcal{S}_{-i}} \end{array}$$

Definition 3.1 imposes a sufficient condition for two types of player i to have the same representation in $\mathcal{S}_i$. The two downward pointing arrows on the right of the diagram coarsen the sigma algebra of every type's beliefs. The commutativity of the diagram then requires the following: If the beliefs of two types θ_i, θ'_i coincide on $K \times \mathcal{S}_{-i}$, then η_i maps θ_i and θ'_i to the same point in $\mathcal{S}_i$. Note that the reverse implication is not required by the diagram. That is, two types with distinct beliefs on $K \times \mathcal{S}_{-i}$ could also be mapped to the same point in $\mathcal{S}_i$.

Thus, in our model, types partition beliefs[1]. This contrasts with Harsanyi

[1] Chen et al. (2016a) introduce the notion of "frames" which are partitions of type spaces that are compatible with the belief structure of the types. Frames are thus a special case of what we call type space quotients.

types spaces, where a type is associated uniquely to a belief, and with universal type spaces, where types and beliefs are homeomorphic.

Definition 3.2 (Strategic Closure). *A family $(\mathcal{A}_i)_i$ of continuous mappings $\alpha_i : \mathcal{S}_i \to \mathcal{B}_i$ is called a* strategically closed family of behaviors *if,*

1. *$\mathcal{A}_i$ contains the constant map equal to A_i*
2. *for every $\alpha_{-i} \in \mathcal{A}_{-i}$, there exists $\alpha_i \in \mathcal{A}_i$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Delta_{K\times\mathcal{S}_{-i}} & \xrightarrow{\psi_i} & \mathcal{S}_i \\ {\scriptstyle id}\downarrow \;\; \downarrow{\scriptstyle \alpha_{-i}} & & \downarrow{\scriptstyle \alpha_i} \\ \Delta_{K\times\mathcal{B}_{-i}} & \xrightarrow{\mathrm{BR}_i} & \mathcal{B}_i \end{array}$$

For a given pair $(\mathcal{S}, \psi)$, a set $\mathcal{A}_i$ consists of correspondences α_i which map points in $\mathcal{S}_i$ to action sets. As a minimality requirement on $\mathcal{A}_i$ point 1 of the definition imposes that each $\mathcal{A}_i$ contains the correspondence $s_i \mapsto A_i$ that precludes no action, for any $s_i \in \mathcal{S}_i$.

In point 2 of the definition, commutativity of the diagram imposes two requirements. First, for a family $\mathcal{A}$ to be strategically closed, the diagram imposes a measurability requirement on $\mathcal{S}$: It requires beliefs that induce different best replies to a behavior in $\mathcal{A}_{-i}$ to be associated to distinct points in $\mathcal{S}_i$. That is, given any profile $\alpha_{-i} \in \mathcal{A}_{-i}$, player i's best-response correspondence to this profile, seen from $\Delta_{K\times\mathcal{S}_{-i}}$ to A_i, is in fact $\mathcal{S}_i$-measurable. Second, any strategically closed family $\mathcal{A}$ must be closed under best replies: A player's best reply to a profile in $\mathcal{A}_i$, viewed as a correspondence from $\mathcal{S}_i$ to A_i is in $\mathcal{A}_i$.

Definition 3.3 (Strategic Type Space (STS)). *A Strategic Type Space (STS) is a type space quotient $(\mathcal{S}_i, \psi_i)_i$ that admits a strategically closed family of behaviors.*

The next definition formalizes the idea that one STS is smaller than another one.

Definition 3.4. *A space $\mathfrak{S} = (\mathcal{S}_i, \psi_i)_i$ is* smaller *than another space $\tilde{\mathfrak{S}} = (\tilde{\mathcal{S}}_i, \tilde{\psi}_i)_i$ if there exist a continuous surjection from $\tilde{\mathcal{S}}$ to $\mathcal{S}$ so that the following diagram commutes:*

$$\begin{array}{ccc} \tilde{\mathcal{S}}_i & \xleftarrow{\tilde{\psi}_i} & \Delta_{K\times\tilde{\mathcal{S}}_{-i}} \\ \downarrow & & \scriptstyle{id}\downarrow \;\; \downarrow \\ \mathcal{S}_i & \xleftarrow[\psi_i]{} & \Delta_{K\times\mathcal{S}_{-i}} \end{array}$$

In this definition a STS is smaller than another STS if the latter admits a representation of the former. That is, all strategic types in the former STS can be obtained by merging strategic types of the latter. The diagram above requires the following sufficient condition for merging types: If the beliefs of types in the latter STS coincide on the smaller STS then these types are merged to the same point in the smaller STS. The definition below then identifies the minimal STS according to Definition 3.4[2].

Definition 3.5 (Minimal STS)**.** *A STS is called minimal if it is smaller than every STS.*

By Definition 3.2, all STS must distinguish types which have different best replies to some strategic behavior. Hence the minimal STS should merge players' types whenever these types have identical best replies to all strategic behaviors from a strategically closed family.

3.3 The Minimal Strategic Type Space

In this section we establish existence and essential uniqueness of the minimal STS. We prove this result by characterizing the minimal STS in terms of ICR hierarchies: First, we show the ICR hierarchies can be recovered from any STS (Lemma 3.1). We then provide a construction of $\mathcal{S}_i$, the set of best reply hierarchies for a game. This construction is canonical as it makes no reference to any Harsanyi type space. We show that these hierarchies coincide with all ICR hierarchies that can arise in all types in all Harsanyi type spaces (Lemma 3.2). We then construct a map ψ_i, which associates beliefs to best-reply hierarchies and prove that the pair $(\mathcal{S}, \psi)$ is a STS (Lemma 3.3). We deduce that $(\mathcal{S}, \psi)$ is a minimal STS and show that it is essentially unique (Theorem 3.1).

Our first theorem states that every STS allows to recover the ICR hierarchies from any Harsanyi type.

[2] Formally, strategic type spaces form a category whose objects are given by the pairs $\mathfrak{S}$ and whose morphisms are given by diagrams as in Definition 3.4. A minimal STS is thus a terminal object in the category.

Lemma 3.1 (STS Factorization of ICR). *For every strategic type space $(\mathcal{S}_i, \psi_i)_i$ and every $m \in \mathbb{N}$, there exists continuous $\alpha_i^m : \mathcal{S}_i \to \mathcal{B}_i$ so that for every Harsanyi type space $(\Theta_i, \pi_i)_i$ and associated maps $(\eta_i)_i$ satisfying the diagram of Definition 3.1,*

$$\mathrm{ICR}_i^m(\theta_i) = \alpha_i^m \circ \eta_i(\theta_i), \ \forall \theta_i \in \Theta_i, \ \forall i \in N$$

The proof of this result, as well as all others, is in the appendix. We denote the ICR correspondence on STS by $\mathrm{ICR}_{\mathcal{S}}(s) := \cap_m \alpha^m(s)$.

We now construct the set $\mathcal{S}$ of all hierarchies of best replies. The first level of the hierarchy is given by a player's best replies to beliefs on nature and any opponents' play. Every subsequent level of a best reply hierarchy is then obtained by computing best replies to beliefs on nature and lower levels of best reply hierarchies.

We construct inductively the sets of m-order best reply hierarchies $\mathcal{S}_i^m$ as m-fold sequences of action set profiles. Let $\mathcal{S}_i^0 := \{A_i\}$ for every i. Given $\mathcal{S}_i^{m-1}$ for every i, we define the subset $\mathcal{S}_i^m \subseteq \mathcal{S}_i^{m-1} \times \mathcal{B}_i$ of sequences of the form $s_m^i = (A_i, b_i^1, \ldots, b_i^m)$ for which there exists a probability distribution $p_i \in \Delta_{K \times \mathcal{S}_{-i}^{m-1}}$ satisfying

$$\mathrm{BR}_i(\mathrm{marg}_{K,l}(p_i)) = b_i^{l+1}, \ \forall \ l < m, \tag{3.1}$$

where $\mathrm{marg}_{K,l}(p_i)$ is the marginal probability of p_i on $K \times \prod_{j \neq i} \mathrm{proj}_l(\mathcal{S}_j^m)$. We define the set of player i's best reply hierarchies as

$$\mathcal{S}_i := \{s_i \in \mathcal{B}_i^{\mathbb{N}} : s_i^m \in \mathcal{S}_i^m, \ \forall \ m \in \mathbb{N}\}.$$

Lemma 3.2 states that the best reply hierarchies $\mathcal{S}$ characterize all ICR hierarchies that can arise in any Harsanyi type space.

Lemma 3.2 (Best-Reply Hierarchies are ICR Hierarchies).

(i) Let $s^m \in \mathcal{B}^m$, then $s^m \in \mathcal{S}^m$ if and only if there exists a Harsanyi type space (Θ, π) and a type profile $\theta \in \Theta$ so that $s^m = (\mathrm{ICR}^l(\theta))_{l \leq m}$.

(ii) Let $s \in \mathcal{B}^{\mathbb{N}}$, then $s \in \mathcal{S}$ if and only if there exists a Harsanyi type space (Θ, π) and a type profile $\theta \in \Theta$ so that $s = (\mathrm{ICR}^l(\theta))_{l \geq 0}$.

For every $m \in \mathbb{N}$, we define a beliefs map $\psi_i^m : \Delta_{K \times \mathcal{S}_{-i}^{m-1}} \to \mathcal{S}_i^m$ by

$$\psi_i^m(p_i) := (A_i, \mathrm{BR}_i(\mathrm{marg}_{K,1}(p_i)), \ldots, \mathrm{BR}_i(\mathrm{marg}_{K,m-1}(p_i))).$$

Any belief p_i on $K \times \mathcal{S}_{-i}$ induces, through the projection on the first m coordinates of $\mathcal{S}_i$, a belief p_i^m on $K \times \mathcal{S}_{-i}^{m-1}$, thus an element $\psi_i^m(p_i^m) \in \mathcal{S}_i^m$. By definition of ψ_i^m, for every $l \leq m$, the first l elements of $\psi_i^m(p_i^m)$ coincide with $\psi_i^l(p_i^l)$. Thus, the sequence $(\psi_i^m(p_i^m))_i$ defines a unique element of $\mathcal{S}_i$, which we denote $\psi_i(p_i)$.

Note that once the set $\mathcal{S}_{-i}$ of all other players' best-reply hierarchies is known, player i's best-reply hierarchies are fully characterized by marginal beliefs and do not depend on correlations across different levels of $\mathcal{S}_{-i}$. Correlations across levels matter in the construction because not all sequences of action-set profiles are in $\mathcal{S}_{-i}$.

We now specify the topology on the set $\mathcal{S}$. Recall that strategic closure requires the minimal STS to admit a closed family of continuous strategic behaviors. By construction, the coordinates of a best-reply hierarchy correspond to a closed family of strategic behaviors. We thus endow $\mathcal{S}$ with its product topology, i.e. the coarsest topology so that all coordinate projections are continuous. Lemma 3.3 below states that $(\mathcal{S}, \psi)$ is a STS. Moreover, the lemma states that $\mathcal{S}$ is a topological quotient of the universal type space of Mertens and Zamir (1985).

Lemma 3.3 (ICR Hierarchies are STS)**.** *$(\mathcal{S}, \psi)$ is a strategic type space. Moreover, the maps η from the universal type space to $\mathcal{S}$ are quotient maps, i.e. continuous open surjections.*

By Lemma 3.1 any finite order ICR hierarchy can be recovered continuously from any STS. By Lemma 3.2 the set $\mathcal{S}$ coincides with all ICR hierarchies. Then by Lemma 3.3, $(\mathcal{S}, \psi)$ is a STS which can be recovered from all STS. The product topology then ensures that $(\mathcal{S}, \psi)$ is a STS which is minimal. As the property of minimality is universal, every minimal STS is homeomorphic to $\mathcal{S}$. Theorem 3.1 thus states existence and essential uniqueness of the minimal STS:

Theorem 3.1 (Existence and Essential Uniqueness of Minimal STS)**.**

(i) *$(\mathcal{S}, \psi)$ is a minimal STS.*

(ii) *If $(\mathcal{S}', \psi')$ and $(\mathcal{S}'', \psi'')$ are minimal STS then $\mathcal{S}''$ and $\mathcal{S}'$ are homeomorphic.*

4 STS-Automaton

The sequence of action sets in a finite order strategic type profile $s^m \in \mathcal{S}^m$ coincides with finite order iterations of ICR and so is decreasing with respect to set inclusion. If an action set is reached by a type at round m and a different action set is reached at round $m+1$, then the former action set will not be reached again. The coordinate transition correspondence $B_i^m : \mathcal{S}_i \rightrightarrows \mathcal{B}_i$ on m-level types is given by $B_i^m(s_i) = \{b_i \in \mathcal{B}_i : (s_i^m, b_i) \in \mathcal{S}_i^{m+1}\}$.

Let $\mathcal{B}_i^m(s_i) := \{\text{proj}_h(s_i) : h \leq m\} \subseteq \mathcal{B}_i$ denote the collection of action sets contained in s_i^m. For every $b_i \in \mathcal{B}_i^m(s_i)$ and let $x_i^m(b_i, s_i) := \#\{h \leq m : \text{proj}_h(s_i) = b_i\}$ be the number of coordinates of s_i^m that contain b_i. For every z and s_i, let the state of the m-order type s_i^m be given by

$$\tau_i^m(s_i) := \{(b_i, x_i^m(b_i, s_i) \mod z) : b_i \in \mathcal{B}_i^m(s_i)\} \tag{4.1}$$

that is, the collection of all action sets contained in s_i^m and the number of coordinates this action set appears modulo z. Let $\mathcal{T}_i^z := \{\tau_i^m(\mathcal{S}_i) : m \in \mathbb{N}\}$ and for every $\tau_i \in \mathcal{T}_i$ let $\bar{b}_i(\tau_i)$ be the smallest action set in $\text{proj}_{\mathcal{B}_i}(\tau_i)$ with respect to set inclusion. The theorem below states that there is a number z so that the profile of transition correspondences of any finite order type depend only on profiles of states.

Lemma 4.1. *There is z and correspondences $(\beta_i : \mathcal{T}_i \rightrightarrows \mathcal{T}_i^z)_i$ so that every i and for all $m \geq z$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{S}_i^m & \overset{B_i^m}{\rightrightarrows} & \mathcal{B}_i \\ {\scriptstyle \tau_i^m}\downarrow & & \uparrow{\scriptstyle \bar{b}_i} \\ \mathcal{T}_i^z & \overset{\beta_i}{\rightrightarrows} & \mathcal{T}_i^z \end{array}$$

For every z, the set $\mathcal{T}^z$ is finite and we may choose the lowest z for which there exists $(\mathcal{T}^z, \beta)$ so that the diagram in Lemma 4.1 commutes for every i and $m \geq z$. Call this minimal pair the STS automaton.

Definition 4.1 (STS Automaton)**.** *The STS automaton is the pair $(\mathcal{T}, \beta) := (\mathcal{T}^z, \beta)$ with the lowest z so that the diagram in Lemma 4.1 commutes for all i and $m \geq z$.*

Let $\mathcal{T}_i^1 = \tau_i^1(\mathcal{S}_i^1)$ be the set of initial states and let the orbit of the STS automaton be sequences of action set profiles whose transitions are determined by the finite correspondence β, i.e. the let

$$\text{Orbit}_\beta(\mathcal{T}^1) := \prod_i \{(\tilde{b}_i(\tau_i^m))_m \in \mathcal{B}_i^{\mathbb{N}} : \tau_i^1 \in \mathcal{T}_i^1, \tau_i^m \in \beta_i(\tau_i^{m-1}),\ \forall\ m\} \quad (4.2)$$

Now the characterization of the STS in terms of the Orbit of the STS automaton is immediate.

Theorem 4.1 (Characterization of Minimal STS)**.** *Let $s_i \in \mathcal{B}_i^{\mathbb{N}}$ for every i, then $s \in \mathcal{S}$ if and only if $s \in \text{Orbit}_\beta(\mathcal{T}^1)$.*

4.1 Examples on $\mathbb{R}^{2\times2\times2}$

In this section we restrict attention to two player $2 \times 2 \times 2$ games. We first illustrate our results and construct the minimal STS in two examples. We then construct the minimal type space for a one-dimensional manifold of payoff structures.

Example 4.1 ($2 \times 2 \times 2$ Coordination Game)**.** Consider the following two-player game: $N = \{1, 2\}$, $K = \{-1, 1\}$ and $A_i = A = \{a, b\}$ where payoffs are given by

	a	b
a	$k,\ k$	$-1,\ 0$
b	$0, -1$	$0,\ 0$

If player i (row player) believes that $k = 1$ with probability less than $\frac{1}{2}$ then b is a dominant action. Otherwise, neither action dominates the other. First order hierarchies of best replies in this game, $\mathcal{S}_i^1$, are thus given by $\{(A, b), (A, A)\}$. The first pair corresponds to beliefs which put less than half of the probability on $k = 1$. Indeed, recall that $\mathcal{S}_{-i}^0 = \{A\}$ from Section 3.3 and consider any belief $p \in \Delta_{K\times\mathcal{S}_{-i}^0}$. Player i thus forms best replies to p-mixtures of state-contingent conjectures $\sigma : K \to \Delta_A$. In the simplex $\Delta_{K\times A}$, these p-mixtures over conjectures form geometric rectangles - the set of probabilities on $K \times A$ with constant marginal belief on K given by (p_1, p_{-1}). The right panel of Figure 1 illustrates these rectangles for $p_1 < \frac{1}{2}$ and $p_1 \geq \frac{1}{2}$. The left of Figure 1 plots the simplex $\Delta_{K\times A}$, where the shaded triangle with dashed contour marks the boundary of the partition induced by the best response correspondence of player i. When $p_1 < \frac{1}{2}$, the mixture of the conjectures is entirely included in the region where b_i is the unique best-response. When $p_1 \geq \frac{1}{2}$, the conjectures cross regions where a and b, or both are best-responses. Hence $\mathcal{S}_i^1 = \{(A, A), (A, b)\}$.

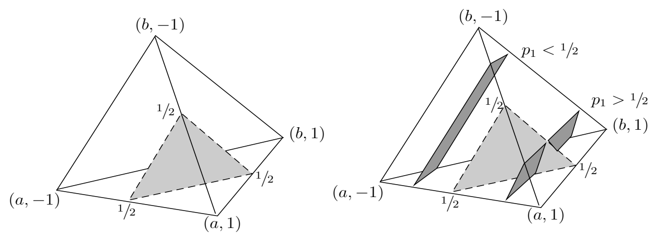

Figure 1: For all beliefs in the region between the shaded triangle (excluded) and the sub simplex spanned by $(a, -1)$, $(b, -1)$ and $(b, 1)$, player i's best response is always b. For all beliefs in the region between the shaded triangle (excluded) and $(a, 1)$, player i's best response is always a and on the shaded triangle all beliefs induce both actions a and b as best response.

We repeat the same procedure on $\mathcal{S}_{-i}^1$. For a belief p on $K \times \mathcal{S}_{-i}^1$ of player i, let p_b denote the probability put on hierarchies ending at b, let p_1 denote the probability put on $k = 1$ and $p_{k,b}$ be the probability joint porbability on state $k \in \{-1, 1\}$ and hierarchies ending at b. As can be seen in Figure 1, $2p_1 - p_b + (p_{-1,b} - p_{1,b}) < 1$ describes the portion of a rectangle associated to p_1 where b is a unique best reply for player i. Hence the set of beliefs on $K \times \mathcal{S}_{-i}^1$ so that BR_i maps to A is given by $p_1 \geq \frac{1}{2}$ and $2p_1 - p_b + (p_{-1,b} - p_{1,b}) \geq 1$. We deduce that $\mathcal{S}_i^2 = \{(A, A, A), (A, A, b), (A, b, b)\}$ corresponds to the following partition of $\Delta_{K \times \mathcal{S}_{-i}^1}$:

(1) $2p_1 - p_b + (p_{-1,b} - p_{1,b}) \geq 1$ and $p_1 \geq 1/2$, for (A, A, A)

(2) $2p_1 - p_b + (p_{-1,b} - p_{1,b}) < 1$ and $p_1 \geq 1/2$, for (A, A, b)

(3) $p_1 < 1/2$, for (A, b, b)

Note that these conditions only depend i's beliefs on K and on the last coordinate in $\mathcal{S}_{-i}^1$. As the last coordinates of $\mathcal{S}^1$ are the same as the last coordinates of $\mathcal{S}^1$ we deduce that the game is indeed simple. Moreover Theorem 4.1 implies that all transitions in $\mathcal{S}$ are described by the three rules above.

Finally, Theorem **??** implies that every type of order $m \geq 2$ can be described by the three transition rules above. The STS automaton in Figure **??** below illustrates the transition for coordinates in $\mathcal{S}_i$ and $\mathcal{S}_{-i}$ in this game:

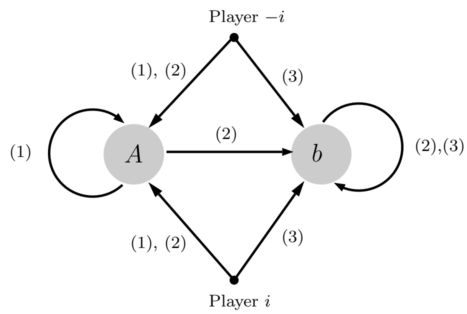

Figure 2: STS Automaton.

The automaton represented in Figure 2 above describes the following transitions: If player i's m-th coordinate is A, then it must be that $p_1 \geq \frac{1}{2}$ and i's $(m+1)$-th coordinate in any strategic type must be one of $\beta_i(A) := \{A, b\}$. Moreover, the $(m+1)$-th coordinate is A if i also believes in strategic types of $-i$ whose m-th coordinate is b with low enough probability. That is, i believes that $-i$ plays b with low enough probability in round m (i.e. condition (1)). Otherwise, the $(m+1)$-th coordinate must be b (condition (2)). However, if i's m-th coordinate was b, then i's $(m+1)$-th coordinate must be $\beta_i(b) = b$. In this case, i's beliefs satisfy either condition (3) or condition (2).

For this example, we can show Corollary 4.1 inductively. Note first that the only possible change in the last coordinate when going from $\mathcal{S}_i^1$ to $\mathcal{S}_i^2$ is to move from A to b. A probability on $K \times \mathcal{S}_{-i}^2$ must therefore put at least as much probability on sequences ending with b than its marginal on $K \times \mathcal{S}_{-i}^1$. But under this constraint, third order types can also only move from A to b or stay unchanged. Hence the automaton above generates all the sequences in $\mathcal{S}$.

In Example 4.2 we consider a game in which the sequence of transition correspondences converges to a cycle.

Example 4.2 (2 × 2 Coordination Game 2)**.** Consider the following two-player game: $N = \{1, 2\}$, $K = \{-1, 1\}$ and $A_i = A = \{a, b\}$ where payoffs are given by

	a	b
a	$k,\ 0$	$-1,-1$
b	$0,\ 0$	$0,\ k$

If player i (row player) believes that $k = 1$ with probability less than $1/2$, then b_i is dominant. If player (column player) $-i$ has such beliefs, a_{-i} is dominant for player $-i$. Hence $\mathcal{S}_i^1 = \{(A,b),(A,A)\}$ and $\mathcal{S}_{-i}^1 = \{(A,a),(A,A)\}$.

For a belief p on $K \times \mathcal{S}_{-i}^1$ of player i, let p_a denote the probability put on hierarchies ending at a and let p_1 denote the probability put on $k = 1$. Then $2p_1 - p_a + (p_{-1,a} - p_{1,a}) < 1$ describes the set of beliefs where a_i is a unique best reply for player i. Hence the set of beliefs on $K \times \mathcal{S}_{-i}^1$ so that BR_i maps to A is given by $p_1 \geq \frac{1}{2}$ and $2p_1 - p_a + (p_{-1,a} - p_{1,a}) \geq 1$. We deduce that $\mathcal{S}_i^2 = \{(A,A,A),(A,A,a),(A,b,b)\}$ corresponds to the following partition of $\Delta_{K\times\mathcal{S}_{-i}^1}$:

(1.1) $2p_1 - p_a + (p_{-1,a} - p_{1,a}) \geq 1$ and $p_1 \geq 1/2$, for (A,A,A)

(2.1) $2p_1 - p_a + (p_{-1,a} - p_{1,a}) < 1$ and $p_1 \geq 1/2$, for (A,A,a)

(3.1) $p_1 < 1/2$, for (A,b,b)

Now let p_b denote the probability put on hierarchies ending at b. Then we obtain $\mathcal{S}_{-i}^2 = \{(A,A,A),(A,A,b),(A,a,a)\}$ as the following partition of $\Delta_{K\times\mathcal{S}_i^1}$:

(1.2) $2p_1 - p_b + (p_{-1,b} - p_{1,b}) \geq 1$ and $p_1 \geq 1/2$, for (A,A,A)

(2.2) $2p_1 - p_b + (p_{-1,b} - p_{1,b}) < 1$ and $p_1 \geq 1/2$, for (A,A,b)

(3.2) $p_1 < 1/2$, for (A,a,a)

Both players prefer to match the opponent's action if $k = 1$. Since players have different dominant actions when $k = -1$, the transition correspondences cycle. Indeed, for m even, $\beta_i^m(A) = \{A,b\}, \beta_{-i}^m(A) = \{A,a\}$, while for m odd, $\beta_i^m(A) = \{A,a\}, \beta_{-i}^m(A) = \{A,b\}$. We illustrate the resulting STS automaton in the figure below.

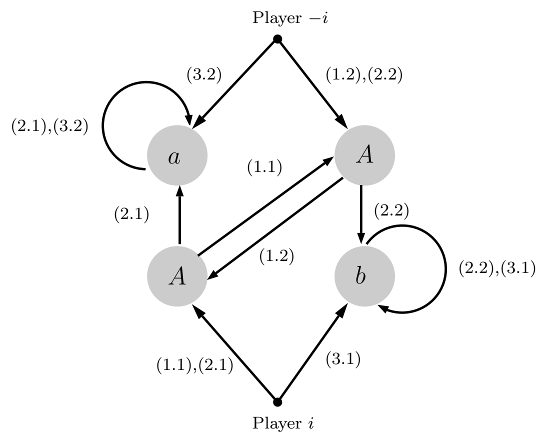

Figure 3: STS Automaton.

5 Robustness to Incomplete Information

A common prior model is a probability measure $P \in \Delta_{K\times\Theta}$ admitting measurable regular conditional probabilities $P_i : \Theta_i \to \Delta_{K\times\Theta_{-i}}$. Common prior models thus give rise to a unique Harsanyi type space. The following defines the class of incomplete information perturbations in line with Kajii and Morris (1997) and Oyama and Takahashi (2020).

Definition 5.1 (ε-elaboration)**.** *An* ε-elaboration *of a complete information game u is a pair $(u^\varepsilon, P^\varepsilon)$, consisting of a payoff structure $u^\varepsilon : K^\varepsilon \times A \to \mathbb{R}^N$ with $k^* \in K^\varepsilon$ and common prior model $P^\varepsilon \in \Delta_{K\times\Theta^\varepsilon}$ so that*

(i) u, u^ε coincide on $\{k^*\} \times A$,

(ii) $P(\tilde{\Theta}^\varepsilon) \geq 1-\varepsilon$, where $\tilde{\Theta}^\varepsilon := \{(k^*, \theta) \in K \times \Theta : \forall\ i,\ P_i(k^*|\theta_i) = 1\}$.

Condition (i) requires the payoff structure of an elaboration to nest the original payoff structure. Condition (ii) requires that type profiles where each player assigns probability one to k^* have probability at least $1-\varepsilon$ under state of nature k^*.

The definition below proposes a definition of robustness to incomplete information for ICR where the class of perturbations of an incomplete information game is given by all ε-elaborations, for ε small enough.

Definition 5.2 (Robustness)**.** *An action set $b_i \subseteq ICR_i(s_i^*)$ is robust to incomplete information if there exists $\overline{\varepsilon} > 0$ so that for every $\varepsilon < \overline{\varepsilon}$ and every ε-elaboration $(u^\varepsilon, P^\varepsilon)$, every $\theta_i^\varepsilon \in \tilde{\Theta}_i^\varepsilon$ satisfies $b_i \subseteq ICR_i(\theta_i^\varepsilon)$.*

A common prior on the STS is a probability distribution $P \in \Delta_{K\times\mathcal{S}}$ so that for all i,

$$\psi_i(P(s_i)) = s_i. \tag{5.1}$$

That is, the entries in s_i must be best-replies to the corresponding marginals of the interim belief $P_i(s_i) := P(\cdot,\cdot|s_i)$. Note that on the support of a common prior there is a one-to-one correspondence between beliefs and strategic types and so STS common priors are also common prior models. We will denote the support of an STS prior on $\mathcal{S}$ by S.

5.1 Extended State Space

Let the *radius* of a type s_i, $\rho_i(s_i)$, be the collection of states reached by s_i, i.e. $\rho_i(s_i) := \{\tau_i^m(s_i) : m \in \mathbb{N}\}$. For each m, let the image of the product map $s \mapsto \overline{\tau}^m(s) := (\tau^m(s), \rho(s))$ be denoted by $\overline{\mathcal{T}}^m := \overline{\tau}^m(\mathcal{S})$ and define the finite set of extended states as $\overline{\mathcal{T}} := \bigcup_m \overline{\mathcal{T}}^m$, which extend the set of state profiles constructed in the previous section by the radius of each type. The lemma below extends Theorem 4.1 to the extended state space $\overline{\mathcal{T}}$

Lemma 5.1. *There exists correspondence $\kappa : \overline{\mathcal{T}} \rightrightarrows \overline{\mathcal{T}}$ so that $\mathcal{S}$ is canonically isomorphic to $\{(t^m)_m \in \overline{\mathcal{T}} : t^1 \in \overline{\mathcal{T}}^1, t^m \in \kappa(t^{m-1}), \forall\, m\}$.*

5.2 Recursive Priors on STS

For any extended state $(\tau_i, r_i) \in T_i$ let $b_i(\tau_i, r_i) \in \mathcal{B}_i$ be the smallest action set (with respect to set inclusion) in $\text{proj}_{\mathcal{B}_i}(\tau_i) \subseteq \mathcal{B}_i$. Call a pair $(T, (p_i : T_i \to \Delta_{K\times T_{-i}})_i)$ STS-closed if p_i is a $\overline{\mathcal{T}}_i$-measurable, single-valued selection of the inverse STS-map ψ_i^{-1}.

Definition 5.3 (STS-closed)**.** *A pair $(T, (p_i)_i)$, where $T \subseteq \overline{\mathcal{T}}$ and $(p_i : T_i \to \Delta_{K\times T_{-i}})_i$, is said to be STS-closed if for every i and $t_i \in T_i$*

(i) $b_i(t_i) = BR_i(p_i(t_i) \circ (id \times b_{-i})^{-1})$

(ii) $t_i' \in \kappa_i|_T(t_i) \implies p_i(t_i) = p_i(t_i') \circ (id \times \kappa_{-i}|_T^{-1})^{-1}$*, where $\kappa|_T$ is the restriction of $t \mapsto (\kappa(t) \cap T)$ to T.*

Definition 5.4 (Minimally STS-closed)**.** *A pair $(T, (p_i)_i)$ is said to be minimally closed if there is no subset $T' \subseteq T$ so that $(T', (p_i|_{T'})_i)$ is STS-closed, where $p_i|_{T'}$ restricts the domain of p_i to T_i.*

For every radius r and subset $T \subseteq \overline{\mathcal{T}}$, let $T(r) := \{(\tau, r') \in T : r' = r\}$ and denote by $d^r_T : T(r) \to \mathbb{N}$ the enumeration of extended states in T satisfying $d^r_T(t) \geq d^r_T(t') \iff t' \in \bigcup_m \kappa^m(t) \cap T$. Let $d_T : T \to \mathbb{N}$ the union of $(d^r_T)_r$. For every m let

$$D^m_T := \{(d_T(t) + m, t_R) : t \in T\},$$

where for every extended state $t \in T$, let t_R recover its radius.

Definition 5.5 (Chain)**.** *A STS-closed pair $(T, (p_i)_i)$ induces a chain that is consistent with a STS prior $P \in \Delta_{K \times \tilde{S}_{-i}}$ if there exists subset $S \subseteq \tilde{S}$ so that for every m and i the diagram below commutes*

$$\begin{array}{ccc} S_i & \xrightarrow{P_i} & \Delta_{K \times S_{-i}} \\ \Big\downarrow \overline{\tau}^m_i & & id \Big\downarrow \quad \Big\downarrow \overline{\tau}^{m-1}_{-i} \\ T_i & \xrightarrow{p_i} & \Delta_{K \times T_{-i}} \end{array}$$

Call S a chain induced by $(T, (p_i)_i)$.

Lemma 5.2. *If a STS-closed pair $(T, (p_i)_i)$ induces a chain $S \subseteq \tilde{S}$ consistent with a STS prior $P \in \Delta_{K \times \tilde{S}_{-i}}$, then there exists an isomorphism $\phi : \bigcup_m D^m_T \cong S$.*

Let $(T, (p_i)_i)$ be STS-closed. For every $t \in T$ and any player i define the overlap

$$T_{-i}(t) := \left\{t' \in T : p(t_i | t'_{-i}) > 0\right\},$$

where $p(t_i|t_{-i}) := \frac{p_j(t'_{-(j,i)} t_i | t'_j)}{\sum_{\tilde{t}_i \in T_i} p_j(t'_{-(j,i)} \tilde{t}_i | t'_j)}$ for any choice of $j \neq i$.

Theorem 5.1. *For every STS-closed pair $(T, (p_i)_i)$, there exists a STS prior P so that $(T, (p_i)_i)$ induces a chain that is consistent with P if and only if there exists $x \in (0,1)^T$ so that for every $t \in T$, x solves the polynomial equality*

$$x_t^{d_T(t)} = \sum_{t' \in T_{-i}(t)} c_i(t, t')\; x_{t'}^{d_T(t')}, \tag{5.2}$$

where $c_i(t, t') := p_i(t_{-i}|t_i) p(t_i | t'_{-i})$.

Lemma 5.3. *Let $(T, (p_i)_i)$ be STS-closed inducing a chain $S \subseteq \tilde{S}$ consistent with STS prior $P \in \Delta_{K\times\tilde{S}}$ and let $x \in [0,1)^T$ satisfy (5.2). Then the restriction of P to the chain S satisfies the following recursive relationship: For every m and $(\delta + m, r) \in D_T^m$,*

$$P(\phi^{-1}(\delta + m, r)) = x_{t(\delta,r)}\ P(\phi^{-1}(\delta + m - 1, r)), \tag{5.3}$$

where $t(\delta, r)$ is the unique extended state with radius r satisfying $d_T^r(t) = \delta$.

5.3 Best-Reply Elaboration and Robustness

A complete information game consists of a payoff structure $u : K \times A \to \mathbb{R}^N$ where $K = \{k^*\}$ is a singleton. The STS of a complete information game is thus also a singleton $\mathcal{S} = \{s^*\}$, where s^* can be represented as the rounds of eliminations of never dominated strategies.

Let $\mathscr{S}_i^0 := 2^{A_i}$, $\Psi_i^1 = BR_i$ and $\mathscr{S}_i^1 = \Psi_i(\Delta_{K\times\mathscr{S}_{-i}^0})$. Then denote the best-reply hierarchies on $\mathscr{S}_i^1$ by $(\mathscr{S}, \Psi)$. Call this the Best-Reply Elaboration of u. Finally let $(\mathscr{T}, \kappa)$ denote the resulting extended state space. Let $\mathfrak{T}$ denote collection of subsets $T \subseteq \mathscr{T}$ so that there exists profile $(p_i : T_i \to \Delta_{\{k^*\}\times T_{-i}})_i$ so that $(T, (p_i)_i)$ is STS closed. Let $\mathcal{P}(T)$ denote the set of such profiles.

Lemma 5.4. *For any $T \in \mathfrak{T}$, the set $\mathcal{P}(T)$ is a convex set with finitely many extreme points $\hat{\mathcal{P}}(T) \subseteq \mathcal{P}(T)$.*

The theorem below characterizes robustness of any finite complete information game in terms of the roots of a finite system of polynomials.

Theorem 5.2 (Robustness)**.** *A complete information game u is not robust to incomplete information if and only if there is $T \in \mathfrak{T}$, $(\hat{p}_i)_i \in \hat{\mathcal{P}}(T)$ and $x \in (0,1)^T$ so that for every t, x_t solves*

$$x_t^{d_T(t)} = \sum_{t' \in T_{-i}(t)} \hat{c}_i(t,t')\ x_{t'}^{d_T(t')},$$

where $\hat{c}_i(t,t') := \hat{p}_i(t_{-i}|t_i)\hat{p}(t_i|t'_{-i})$.

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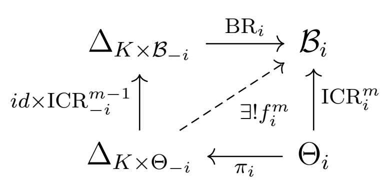
A Appendix

A.1 Existence and Uniqueness of minimal STS

We introduce some additional notation. For any mapping $f : X \to Y$ we denote the image of f by $\mathrm{Im}(f)$.The following lemma is key for our results:

Lemma A.1 (BR Factorization of ICR)**.** *Let $(\Theta_i, \pi_i)_i$ be a Harsanyi type space. Then for every m and every i, ICR_i^m admits the following factorization through a unique f_i^m,*



Proof. The part of the diagram that is not trivial is the upper left corner. Let $\sigma : K \times \Theta_{-i} \to \Delta_{A_{-i}}$ be a $\pi_i(\theta_i)$-measurable conjecture. Write the θ_i mixture of σ as

$$\langle \sigma, \pi_i(\theta_i)\rangle_{\Theta_i}(k, a_{-i}) := \int_{\Theta_{-i}} \sigma(k, \theta_{-i})(a_{-i})\ \pi_i(\theta_i)(k, \mathrm{d}\theta_{-i}),\ \forall\ k, a_{-i}$$

Then by definition of ICR we have that

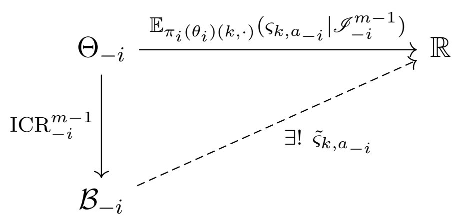
$$\mathrm{ICR}_i^m(\theta_i) = \left\{\mathbf{B}(\langle\sigma, \theta_i\rangle_{\Theta_i}) : \sigma \text{ is } \pi_i(\theta_i)\text{-meas., supp } \sigma(k, \theta_{-i}) \subseteq \mathrm{ICR}_{-i}^{m-1}(\theta_{-i})\right\}$$

where $\mathbf{B}(p) := \mathrm{argmax}_{a_i} \sum_{k,a_{-i}} u_i(k, a_i, a_{-i})p(k, a_{-i})$ for every $p \in \Delta_{K\times A_{-i}}$. We now show that for every $\pi_i(\theta_i)$-measurable conjecture $\sigma : K \times \Theta_{-i} \to$

$\Delta_{A_{-i}}$ so that supp $\sigma(k, \theta_{-i}) \subseteq \mathrm{ICR}^{m-1}_{-i}(\theta_{-i})$ we can construct a conjecture $\tilde{\sigma} : K \times \mathcal{B}_{-i} \to \Delta_{A_{-i}}$ so that supp $\sigma(k, b_{-i}) \subseteq b_{-i}$ and

$$\langle \tilde{\sigma}, p_i^{m-1}(\theta_i)\rangle(k, a_{-i}) = \langle \sigma, \pi_i(\theta_i)\rangle_{\Theta_i}(k, a_{-i}), \ \forall \ k, a_{-i}$$

where $p_i^{m-1}(\theta_i) := \pi_i(\theta_i) \circ (id \times \mathrm{ICR}^{m-1}_{-i})^{-1}$ is the push forward probability on the left of the diagram above. Let $\mathscr{I}^{m-1}_{-i}$ be the sigma algebra generated by ICR^{m-1}_{-i} and write $\varsigma_{k,a_{-i}} : \theta_{-i} \mapsto \sigma(k, \theta_{-i})(a_{-i})$. Consider the conditional expectation $\mathbb{E}_{\pi_i(\theta_i)}(\varsigma_{k,a_{-i}} | \mathscr{I}^{m-1}_{-i}) : \Theta_{-i} \to [0, 1]$ and define for every k, a_{-i} our conjecture $\tilde{\sigma}(a_{-i}) : (k, b_{-i}) \mapsto \tilde{\varsigma}_{k,a_{-i}}(b_{-i})$ from the unique mapping $\tilde{\varsigma}_{k,a_{-i}} : \mathcal{B}_i \to [0, 1]$ so that the diagram below commutes



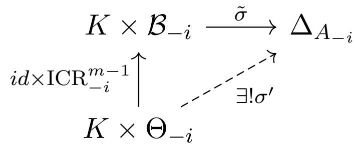
Then, by the defining property of conditional expectation and the commutativity of the last diagram we have that for every k, a_{-i}

$$\begin{aligned}\langle \sigma, \pi_i(\theta_i)\rangle_{\Theta_i}(k, a_{-i}) &= \int_{\Theta_{-i}} \mathbb{E}_{\pi_i(\theta_i)(k,\cdot)}(\varsigma_{k,a_{-i}} | \mathscr{I}^{m-1}_{-i})(\theta_{-i}) \ \pi_i(\theta_i)(k, \mathrm{d}\theta_{-i}) \\ &= \sum_{b_{-i} \in \mathcal{B}_{-i}} \tilde{\sigma}(k, b_{-i})(a_{-i}) \ p_i^{m-1}(\theta_i)(k, b_{-i})\end{aligned}$$

We deduce that for every conjecture σ on types in Θ_{-i} there exists a conjecture $\tilde{\sigma}$ which is constant on the fibers of ICR^{m-1}_{-i} so that

$$\mathbf{B}(\langle \tilde{\sigma}, p_i^{m-1}(\theta_i)\rangle) = \mathbf{B}(\langle \sigma, \pi_i^{m-1}(\theta_i)\rangle_{\Theta_{-i}})$$

Which implies that every m order rationalizable action of a type θ_i is also an action contained in the image of $\mathrm{BR}_i(p_i^{m-1}(\theta_i))$. To show that this image is not strictly larger than $\mathrm{ICR}^m_i(\theta_i)$, note that every conjecture $\tilde{\sigma} : K \times \mathcal{B}_{-i} \to \Delta_{A_{-i}}$ so that supp $\tilde{\sigma}(k, b_{-i}) \subseteq b_{-i}$ also induces a $\pi_i(\theta_i)$-measurable conjecture $\sigma' : K \times \Theta_{-i} \to \Delta_{A_{-i}}$ which respects the support condition. This conjecture is defined uniquely through the diagram below:



∎

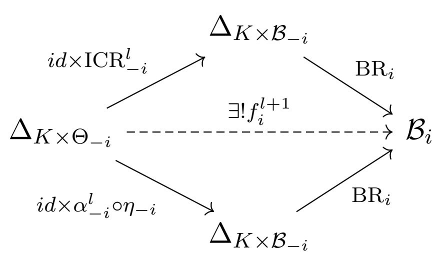
Lemma A.2 (STS Factorization of ICR)**.** *For every strategic type space $(\mathcal{S}_i, \psi_i)_i$ and every $m \in \mathbb{N}$, there exists continuous $\alpha_i^m : \mathcal{S}_i \to \mathcal{B}_i$ so that for every Harsanyi type space $(\Theta_i, \pi_i)_i$ and associated maps $(\eta_i)_i$ satisfying the diagram of Definition 3.1,*

$$\mathrm{ICR}_i^m(\theta_i) = \alpha_i^m \circ \eta_i(\theta_i), \ \forall \theta_i \in \Theta_i, \ \forall i \in N$$

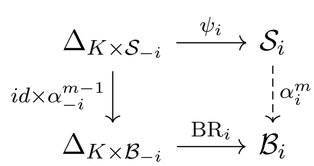
Proof. We need to show that $\theta \mapsto (\mathrm{ICR}^m(\theta))_m$ is constant on the fibers of η, which we prove by induction. The base case is trivial as it is given by the constant function $\theta \mapsto \mathrm{ICR}^0(\theta) = \{A\}$. Suppose now that $\theta \mapsto (\mathrm{ICR}^l(\theta))_{l \leq m-1}$ is constant on the fibers of η. Then by inductive hypothesis there exist $\alpha_i^l : \mathcal{S}_i \to \mathcal{B}_i$ so that for every i and $l \leq m-1$ we have

$$\mathrm{ICR}_i^l(\theta_i) = \alpha_i^l \circ \eta_i(\theta_i)$$

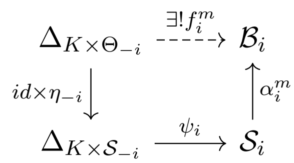
But then the following diagram must also commute for every $l \leq m-1$:



Let $l = m-1$. By Lemma A.1 the uniquely defined mapping $f_i^m : \Delta_{K \times \Theta_{-i}} \to \mathcal{B}_i$ induces ICR_i^m, i.e. $f_i^m \circ \pi_i = \mathrm{ICR}_i^m$. By the strategic closure property of strategic types we also have that there exists $\alpha_i^m : \mathcal{S}_i \to \mathcal{B}_i$ so that



Putting the last two diagrams together thus shows that f_i^m can also be factorized as



Finally recall that by the type space quotient property of $\mathcal{S}$ we have that $f_i^m \circ \pi_i = \alpha_i^m \circ \eta_i$. As each diagram pins down f_i^m uniquely, we deduce that $\mathrm{ICR}_i^m = \alpha_i^m \circ \eta_i$, as required.

□

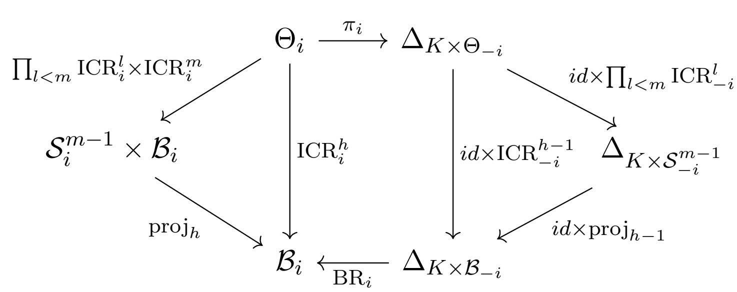
Lemma A.3 (Best-Reply Hierarchies are ICR Hierarchies)**.**

(i) Let $s^m \in \mathcal{B}^m$, then $s^m \in \mathcal{S}^m$ if and only if there exists a Harsanyi type space (Θ, π) and a type profile $\theta \in \Theta$ so that $s^m = (\mathrm{ICR}^l(\theta))_{l\leq m}$.

(ii) Let $s \in \mathcal{B}^{\mathbb{N}}$, then $s \in \mathcal{S}$ if and only if there exists a Harsanyi type space (Θ, π) and a type profile $\theta \in \Theta$ so that $s = (\mathrm{ICR}^l(\theta))_{l\geq 0}$.

Proof. We prove each point in turn:

(i) We prove the "if" part of (i) inductively. Let $(\Theta_i, \pi_i)_i$ be a Harsanyi type space. The base case $m = 0$ is trivial. We thus proceed directly to the inductive step, where by inductive hypothesis on $m-1$ the diagram below commutes for every $h \in \{1, \dots, m-1\}$.



Indeed, the inductive hypothesis states that for every Harsanyi type there exists a belief $p \in \Delta_{K\times\mathcal{S}_{-i}^{m-2}}$ so that $\mathrm{ICR}_i^h(\theta_i) = \mathrm{BR}_i(\mathrm{marg}_{h-1}(p))$ for every $h < m$. The belief p is obtained by the arrow which goes south east from $\Delta_{K\times\Theta_{-i}}$ and then we follow the subsequent arrows down to $\mathcal{B}_i$. For $h = m$, the inductive hypothesis ensures commutativity of the right triangle and the rest follows from Lemma A.1, i.e. the factorization of ICR_i^m.

We now prove the converse constructively by induction. For the base case $m = 1$, add one first order belief for player i given by $p_i^1 = \mathrm{marg}_K(p_i^m)$ and take any Harsanyi type space which contains a type of i with this first order belief, i.e. with $\mathrm{marg}_K(\pi_i(\theta_i)) = p_i^1$. We proceed with the inductive step. For $m > 1$ and for every i fix a belief $p_i^m \in \Delta_{K\times\mathcal{S}_{-i}^{m-1}}$ and denote $p_i^{m-1} := \mathrm{marg}_{K\times\mathcal{S}_{-i}^{m-2}}(p_i^m)$. By inductive hypothesis, we may suppose that there exists a Harsanyi type space $(\Theta_i, \pi_i)_i$ so that for every $s_{-i}^{m-1} \in \mathrm{supp}\ \mathrm{marg}_{\mathcal{S}_{-i}^{m-1}}(p_i^m)$ there exists θ_{-i}^{m-1} (identified directly as $m-1$ order beliefs) so that $(\mathrm{ICR}_{-i}^l(\theta_{-i}))_{l=0}^{m-1} = s_{-i}^{m-1}$. By inductive hypothesis we have that for every $l < m-1$ and we have found $(\Theta_i^l)_{l<m-1}$ so that the diagram below commutes.

$$
\begin{array}{ccccc}
\Theta_i^{l+1} & \hookrightarrow & \Delta_{K\times\Theta_{-i}^l} & \xrightarrow{id\times\prod_{q<l+1}\mathrm{ICR}_{-i}^q} & \Delta_{K\times\mathcal{S}_{-i}^l} \\
{\scriptstyle \mathrm{marg}_{K\times\Theta_{-i}^{l-1}}}\big\downarrow & & \big\downarrow{\scriptstyle id\times\prod_{j\neq i}\mathrm{marg}_{K\times\Theta_{-j}^{l-2}}} & & \big\downarrow{\scriptstyle id\times\mathrm{proj}_{\mathcal{S}_{-i}^{l-1}}} \\
\Theta_i^l & \hookrightarrow & \Delta_{K\times\Theta_{-i}^{l-1}} & \xrightarrow{id\times\prod_{q<l}\mathrm{ICR}_{-i}^q} & \Delta_{K\times\mathcal{S}_{-i}^{l-1}}
\end{array}
$$

Moreover, for $l = m-1$, the right square of the diagram commutes. Since both vertical arrows on the right square are projections applied on the support of beliefs, the dashed arrow is indeed given by the marginal probability operator. There is thus $\Theta_i^m \subseteq \Delta_{K\times\Theta_{-i}^{m-1}}$ which makes the entire diagram commute for $l = m-1$. Making choices up to each order m yields a suitable Harsanyi type space for every m. We have thus shown the result for hierarchies of beliefs.

(ii) The diagram above can be extended to that of an inverse system. The limit of this system uniquely characterizes topological spaces $(\mathcal{S}_i)_i$ and continuous maps $\psi_i : \Delta_{K\times\mathcal{S}_{-i}} \to \mathcal{S}_i$ so that the upper part of the above diagram commutes with the lower part for every m.

$$
\begin{array}{ccccccc}
\Theta_i & \overset{\pi_i}{\dashrightarrow} & \Delta_{K\times\Theta_{-i}} & \xrightarrow{id\times\prod_{m\in\mathbb{N}}\mathrm{ICR}^m_{-i}} & \Delta_{K\times\mathcal{S}_{-i}} & \overset{\psi_i}{\dashrightarrow} & \mathcal{S}_i \\
{\scriptstyle \mathrm{marg}_{K\times\Theta^{m-1}_{-i}}}\downarrow & & \downarrow{\scriptstyle id\times\mathrm{marg}_{K\times\Theta^{m-2}_{-i}}} & & \downarrow{\scriptstyle id\times\mathrm{proj}_{m-1}} & & \downarrow{\scriptstyle \mathrm{proj}_{\mathcal{S}^m_i}} \\
\Theta^m_i & \hookrightarrow & \Delta_{K\times\Theta^{m-1}_{-i}} & \xrightarrow{id\times\prod_{l<m}\mathrm{ICR}^l_{-i}} & \Delta_{K\times\mathcal{S}^{m-1}_{-i}} & \xrightarrow{\psi^m_i} & \mathcal{S}^m_i \\
{\scriptstyle \mathrm{marg}_{K\times\Theta^{m-2}_{-i}}}\downarrow & & \downarrow{\scriptstyle id\times\mathrm{marg}_{K\times\Theta^{m-3}_{-i}}} & & \downarrow{\scriptstyle id\times\mathrm{proj}_{m-2}} & & \downarrow{\scriptstyle \mathrm{proj}_{\mathcal{S}^{m-1}_i}} \\
\Theta^{m-1}_i & \hookrightarrow & \Delta_{K\times\Theta^{m-2}_{-i}} & \xrightarrow{id\times\prod_{l<m-1}\mathrm{ICR}^l_{-i}} & \Delta_{K\times\mathcal{S}^{m-2}_{-i}} & \xrightarrow{\psi^{m-1}_i} & \mathcal{S}^{m-1}_i
\end{array}
$$

Now we may choose each $\Theta^m_i \hookrightarrow \Delta_{K\times\Theta^{m-1}_{-i}}$ to be an equality instead of a strict inclusion. Then by Theorem 1 in Part III of Mertens, Sorin, and Zamir (2015) the resulting inverse limit $(\Theta_i, \pi_i)_i$ is the universal type space with π_i a homeomorphism. Applying (i) for the universal type space and using the fact that ICR depends only on universal types (as shown in Proposition 1 of Dekel et al. (2007)) yields the result.

□

Theorem A.1 (Existence and Essential Uniqueness of Minimal STS)**.**

(i) $(\mathcal{S}, \psi)$ *is a minimal STS.*

(ii) If $(\mathcal{S}', \psi')$ *and* $(\mathcal{S}'', \psi'')$ *are minimal STS then* $\mathcal{S}''$ *and* $\mathcal{S}'$ *are homeomorphic.*

(iii) The maps η *from the universal type space to* $\mathcal{S}$ *are quotient maps, i.e. continuous open surjections.*

Proof. We already characterized the map ψ_i in (ii). Lemma 3.1 showed that a STS must embed all possible ICR sequences arising from any Harsanyi type space. Let $(\mathcal{S}^*, \psi^*)$ be the minimal STS. Then it has to be the case that the set $\mathcal{S}$ can be injected in to the set $\mathcal{S}^*$. Then note that $\mathcal{S}$ satisfies the strategic closure property, where the family $\mathcal{A}$ is given by the coordinate projections. Moreover, the coarsest topology on $\mathcal{S}$ that makes each $\alpha \in \mathcal{A}$ continuous (as required by the strategic closure property) is the coarsest topology which make the projections continuous, i.e. the product topology. By the universal property of the product topology there exists a unique homeomorphism $\mathcal{S}^* \cong \mathcal{S}$. Since we constructed a continuous ψ_i by endowing the inverse limit $\mathcal{S}_i$ with the product topology, $(\mathcal{S}, \psi)$ is the minimal STS, where the mappings $\eta : \Theta \to \mathcal{S}$ are those described in (ii) by taking the quotient of types and beliefs with respect to the ICR hierarchies. Hence uniqueness up to homeomorphisms follows from the universal property of the product topology in the category of topological spaces. In particular, we deduce that the maps $\eta_i = (\mathrm{ICR}_i^m)_m$ seen as continuous maps from the universal type space to the topological space $\mathcal{S}_i$ characterized above are quotient maps and so are open. Indeed, the product topology on universal types is generated by cylinders which are mapped to cylinders in $\mathcal{S}$. Hence every open set is mapped to an open set by η. □

A.2 STS-Automaton

Theorem A.2. *There is z and a correspondence $\beta : \mathcal{P}_z \rightrightarrows \mathcal{P}_z$ so that for all $m \geq z$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{S}^m & \overset{B^m}{\rightrightarrows} & \mathcal{B} \\ {\scriptstyle \tau_z^m}\downarrow & & \uparrow{\scriptstyle \overline{b}} \\ \mathcal{T}_z & \overset{\beta}{\rightrightarrows} & \mathcal{T}_z \end{array}$$

Proof. Let $\mathcal{T} := \mathcal{S} \times \mathcal{B}$ and define the operator $T : 2^{\mathcal{T}} \to 2^{\mathcal{T}}$ as follows

$$T(R) := \prod_i \{(\psi_i(\mathrm{marg}_{K\times\mathcal{S}_{-i}}(q_i)), \mathrm{BR}_i(\mathrm{marg}_{K\times\mathcal{B}_{-i}}(q_i))) : q_i \in \Delta_{K\times R_{-i}}\}$$

For any $s^h \in \bigcup_m \mathcal{S}^m$ let $\max \mathcal{S}(s^h)$ denote the set of types $\overline{s}$ so that $\overline{s}^h = s^h$ whose limit $\lim_m \mathrm{proj}_m(\overline{s})$ is maximal in $\lim_m \mathrm{proj}_m(\mathcal{S})$ with respect to set inclusion.

Let $\overline{S}^1 := \{\max \mathcal{S}(s^1) : s^1 \in \mathcal{S}^1\}$. Note that this set is finite by the monotonicity of coordinate transitions. Let $R := \{(s, s^1) : s \in \overline{S}^1\}$ and for any h let $T^{1,h}(R) := T^h(R) \cap (\overline{S}^1 \times \mathcal{B})$, where $T^h = T \circ \cdots \circ T$ is a h-fold application of T. Given $\overline{S}^{m-1}$ and $(T^{m-1,h})_h$, define $\overline{S}^m := \bigcup_h \{\max \mathcal{S}(s^h, b) : (s, b) \in (T^{m-1})^h(R)\}$. Note that $\overline{S}^{m-1} \subseteq \overline{S}^m$.

Let $Q_1^{m,h}(R) := T(T^{m-1,h-1}(R)) \cap (\overline{S}^m \times \mathcal{B})$. We call this h-level, round 1 best-replies. The operator computes h-level best replies of types in $\overline{S}^m$ to beliefs with support on $(h-1)$-level best replies of types in $\overline{S}^{m-1}$. Given $h-1$-level, round $l-1$ best-replies, $Q_{l-1}^{m,h-1}$, define $Q_l^{m,h}(R) := T(Q_{l-1}^{m,h-1}(R)) \cap (\overline{S}^m \times \mathcal{B})$. This computes h-level best replies of types in $\overline{S}^m$ to beliefs with support on $(h-1)$-level, round $l-1$ best-replies of types in $\overline{S}^m$. The collection of best replies in round l at each level of a given type in $\overline{S}^m$ nests its best replies in round $l-1$. I.e. $Q^{m,h}_{l-1}(R) \subseteq Q^{m,h}_{l}(R)$. Indeed, since $\overline{S}^{m-1} \subseteq \overline{S}^m$, we have that $Q^{m,h}_1(R) \subseteq Q^{m,h}_2(R)$. Deduce that for all m, x, $\min\{l \in \mathbb{N} : Q^{m,x}_l = Q^{m,x}_{l+1}\} \leq 2^{\#\mathcal{B}}$ and so there exists $M_m \leq \#2^{\mathcal{B}}$ so that $Q^{m,h}_{M_m}(R) = Q^{m,h}_{M_m+x}(R)$ for all $x > 0$. Finally, set $T^{m,h}(R) := Q^{m,h}_{M_m}(R)$. Similarly, note that for every h, $T^{m-1,h}(R) \subseteq T^{m,h}(R)$.

We now show that for any h' large enough and any $h > h'$ large enough, the restriction of $T^{m,h}$ to types which have converged at level h' is cyclic.

First, note that $T^{1,h}(R)$ is cyclic, i.e. there is bounded $\tilde{z}, y_1 \leq \#(\overline{S}^1 \times \mathcal{B})$ so that for all $h \geq y_1$, $T^{1,h}(R) = T^{1,h+\tilde{z}}(R)$. This cyclicality is then transmitted to all operators constructed above: For every m, h let $\overline{\mathcal{S}}^{m,h} := \{s \in \overline{\mathcal{S}}^m : \mathrm{proj}_h(s) = \lim_l \mathrm{proj}_l(s)\}$. For $y_{m-1} \leq h' \leq h - \tilde{z}$, if $T^{m-1,h}(R) \cap (\overline{\mathcal{S}}^{m-1,h'} \times \mathcal{B}) = T^{m-1,h+\tilde{z}}(R) \cap (\overline{\mathcal{S}}^{m-1,h'} \times \mathcal{B})$, then for $y_{m,1} \leq h' \leq h - \tilde{z}$, $Q^{m,h}_1(R) \cap (\overline{\mathcal{S}}^{m,h'} \times \mathcal{B}) = Q^{m,h+\tilde{z}}_1(R) \cap (\overline{\mathcal{S}}^{m,h'} \times \mathcal{B})$, where $y_{m,1} = y_{m-1} + 1$. So for all $h \geq y_m := y_{m-1} + M_m$ we have $T^{m,h}(R) = T^{m,h+\tilde{z}}(R)$.

By the monotonicity of $T^{m,h}$ with respect to set inclusion and the cyclicality, deduce that there is $M \leq \prod_{j=0}^{\#\mathcal{B}} \#\overline{S}^1 \tilde{z}(\#\mathcal{B} - j)$ so that for all h, $T^{M,h}(R) = T^{M+1,h}(R)$. Indeed, each type in $\overline{S}^1$ has at most $\tilde{z}$ many distinct transitions, each transition is limited to at most $\#\mathcal{B}$ different action sets. Each such transition then corresponds to a type that can again have at most $\tilde{z}$ distinct transitions, where each such transition is limited to at most $\#\mathcal{B} - 1$ different action set. Finally let $T^h(R) := T^{M,h}(R)$ and so there is $\underline{m}$ so that $T^h(R) \cap (\overline{\mathcal{S}}^{m,h'} \times \mathcal{B}) = T^{h+\tilde{z}}(R) \cap (\overline{\mathcal{S}}^{m,h'} \times \mathcal{B})$ for all $h - \tilde{z} \geq h' \geq \underline{m}$.

Let $z := \underline{m}\tilde{z}$ and deduce that for every types $s, \tilde{s}$ and $m, \tilde{m}$, if $\mathcal{B}(s^m) = \mathcal{B}(\tilde{s}^{\tilde{m}})$ and for all $b \in \mathcal{B}(s^m)$ and $x^m_z(b, s) = x^{\tilde{m}}_z(b, \tilde{s})$ then $T^m(R) \cap \{s\} \times \mathcal{B} = T^{\tilde{m}}(R) \cap \{\tilde{s}\} \times \mathcal{B} \iff B^m(s^m) = B^{\tilde{m}}(\tilde{s}^{\tilde{m}})$. □

Lemma A.4. *(i) For every $s \in \mathcal{S}$ and every $m \in \mathbb{N}$, the family of cylinder sets*

$$\mathcal{O}^m(s) := \left\{O^m := \{\tilde{s} \in \mathcal{S} : \tilde{s}^l = s^l\} : l \geq m\right\}$$

is a neighborhood base of s.

(ii) There exists m so that the last coordinate of the truncated sequence s^m reaches an absorbing node in $\mathfrak{N}$ if and only if $\{s\}$ is open and closed in $\mathcal{S}$, i.e. an isolated point.

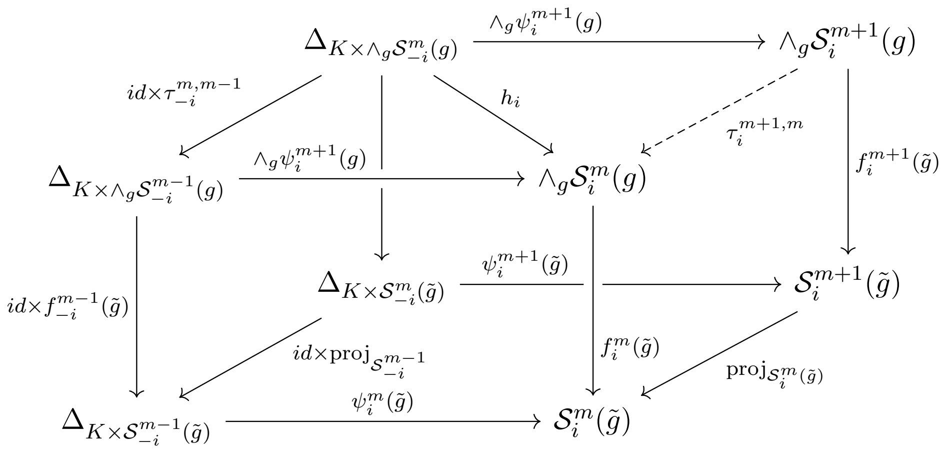
(iii) $\mathcal{S}$ is compact and Hausdorff.

Proof. We know that $\mathcal{S}$ is endowed with its product topology. Then (i) is immediate from Theorem 4.1. We thus immediately deduce that $\{s\}$ is open if and only if its path converges to an absorbing node in $\mathfrak{N}$ after finite m. For any non critical type the Hausdorff property is trivial. The only problematic case would be critical types s, s' who share the same path up to some finite m. However s, s' can also be separated by neighborhoods since $s^m \neq s'^m$ implies $\mathcal{O}^m(s') \cap \mathcal{O}^m(s) = \varnothing$. We deduce that $\mathcal{S}$ is Hausdorff. Then (ii) follows from the Hausdorff property. Indeed, $\mathcal{S}$ being Hausdroff implies that $\{s\}$ is closed for all $s \in \mathcal{S}$ and so s is non critical if and only if $\{s\}$ is open and closed and so a connected component. To finish proving (iii) let $\mathcal{U}$ be an open cover of $\mathcal{S}$. Then for every critical type s^*, there exists an open set $U(s^*) \in \mathcal{U}$ so that $s^* \in U(s^*)$. By (i), there exists $m(s^*) \in \mathbb{N}$ and $O^{m(s^*)} \in \mathcal{O}^{m(s^*)}(s^*)$ so that $O^{m(s^*)} \subseteq U(s^*)$. Let $\mathcal{C}$ be the the set of non cyclic paths in $\mathfrak{N}$ which go from A and end at a node which has an edge pointing to itself. We conclude that there exists a subfamily of open sets $\mathcal{U}^* \subseteq \mathcal{U}$ isomorphic to $\mathcal{C}$ so that the union of its members covers all critical types. Since $\mathcal{C}$ is finite, the family $\mathcal{U}^*$ is finite and covers all critical types. But note that the set $\mathcal{S} \setminus \bigcup_{U^* \in \mathcal{U}^*} U^*$ is open (by (ii)) and finite. Hence $\mathcal{U}$ admits a finite subcover. □

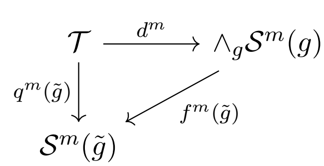
A.3 Minimal Strategic Type Spaces for Classes of Games

Theorem A.3 (Existence and Uniqueness of Minimal G-STS)**.** *The join induces a unique inverse limit $(\bigwedge_{g \in G} \mathcal{S}(g), \bigwedge_{g \in G} \psi(g), (f(g))_g)$ which is the minimal G-STS. Moreover, the minimal G-STS is unique up to isomorphisms.*

Proof. First note that the sequence $(\wedge_g \mathcal{S}_i^m(g), \wedge_g \psi_i^m(g), \wedge_g \mathcal{S}_{-i}(g))_m$ is a well defined inverse system and so admits an inverse limit. Indeed, an inductive argument shows that for all m, there exist unique $(\tau_i^{m+1,m})_i$ so that for all $\tilde{g} \in G$ the following diagram commutes:



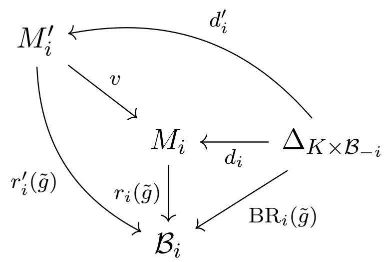
For $m = 1$, $\tau_{-i}^{1,0}$ is just the constant map for all $\tilde{g} \in G$. Hence there exists a unique map $\tau_i^{m+1,m}$ so that the diagram commutes for all $\tilde{g} \in G$. To see this, consider only the top face of the cube. For given $\tau_{-i}^{m,m-1}$ there exists a unique $\tau_i^{m+1,m}$ so that the top face commutes. Indeed, fix any $s \in \wedge_g\mathcal{S}_i^m(g)$ and suppose there is $p, p' \in \Delta_{K\times\wedge_g\mathcal{S}^m_{-i}(g)}$ so that $p \in h_i^{-1}(s)$ and $p' \notin h_i^{-1}(s)$ but $\wedge_g\psi_i^{m+1}(g)(p) = \wedge_g\psi_i^{m+1}(g)(p')$. Then for all $l \leq m+1$ and $g \in G$ we also have $f_i^{m+1}(\tilde{g})(\wedge_g\psi_i^{m+1}(g)(p)) = f_i^{m+1}(\tilde{g})(\wedge_g\psi_i^{m+1}(g)(p'))$. By minimality of $\wedge_g\mathcal{S}_i^m(g)$ it thus has to be the case that $h_i(p) = h_i(p')$, a contradiction. Hence $\tau_i^{m+1,m}$ exists. But then the entire cube commutes since commutativity of all paths not involving $\tau_i^{m+1,m}$ is given by definition of the join. Hence $(\tau^{m+1,m}, \wedge_g\mathcal{S}^m)_m$ define an inverse system with an inverse limit. Then note that any minimal G-STS $(\mathcal{T}, q)$ must admit a continuous surjection d^m for every m so that for every $\tilde{g} \in G$ the following commutes:



Then by the universal property of the inverse limit we deduce that any minimal G-STS is thus isomorphic to $(\wedge_g\mathcal{S}(g), \wedge_g\psi(g))$. $\square$

Lemma A.5 (Manifold Representation of Minimal G-STS). *Let $G \subseteq \mathbb{R}^{A\times K\times N}$ be a finite dimensional manifold. Then there exists a manifold M, an immersion $h : M \to G$ and a correspondence $\beta_G : M \rightrightarrows M$ so that the set $\{(r^m)_m : r^{m+1} \in \beta_G(r^m),\ \forall\ m\}$ is isomorphic to $\bigwedge_{g\in G} \mathcal{S}(g)$.*

Proof. A factorization of $(\mathrm{BR}_i(g))_g$ is a set M_i' and mappings $r' = (r_i'(g) : M_i' \to \mathcal{B}_i)_g, d_i' : \Delta_{K\times\mathcal{B}_{-i}} \to M_i'$ so that $\mathrm{BR}_i(g) = r_i(g) \circ d$ for all g. Let (M, r, d) be the unique factorization of $(\mathrm{BR}_i(g))_g$ so that for any other factorization (M', r', d') there exists a unique surjection v so that the diagram below commutes



M_i is thus isomorphic to the join of all pre-images $(\mathrm{BR}_i^{-1}(g)(\mathcal{B}_i))_g$. Let (U, ϕ) be a manifold chart of G. By upper-hemi-continuity of best replies, for every $g \in U$ there exists a neighborhood $O(g)$ of g and a set $\mathcal{O}(g) \subseteq \mathcal{B}_i$ so that $\mathcal{O}(g) = \mathrm{Im}(\mathrm{BR}_i(g'))$ for all $g' \in O(g)$. Fix any $b_i \in \mathcal{O}(g)$ that is not maximal in $\mathcal{O}(g)$ with respect to set inclusions. For every $g' \in O(g)$, there exists a hyperplane $H(b_i, g')$ which separates $\{p : \mathrm{BR}_i(g')(p) \subseteq b_i\}$ and $\{p : b_i \subsetneq \mathrm{BR}_i(g')(p)\}$. For each such non-maximal $b_i \in \mathcal{O}(g)$, the collection of hyperplanes $(H(b_i, g'))_{g'\in O(g)}$ is isomorphic to $W(b_i) := \cup_{g'\in O(g)} d_i(H(b_i, g'))$. Moreover, the collection $(H(b_i, g'))_{g'\in O(g)}$ corresponds to an open set in G and so induces a chart on $W(b_i)$ and an immersion into $O(g)$ given by $\varphi(d_i(H(b_i, g'))) = g'$. Hence $\phi \circ \varphi$ thus defines a chart on $W(b_i)$ for non maximal action set b_i. If b_i is maximal, then $d_i(\{p : b_i \subsetneq \mathrm{BR}_i(g')(p)\})$ is mapped to an isolated point in M_i and so can be immersed in $O(g)$. ☐

A.4 STS Characterization of Robustness to Incomplete Information